EMBRACING PROVING INTO EVERYDAY LESSON BY PROBLEM POSING

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Proof plays significant roles in the context of school mathematics and is a tool for enhancing student's understanding of mathematics. Lack of opportunities for proving in textbook has been documented. This study was conducted to consider an instructional way to make proving as everyday lesson by formulating more opportunities than did textbooks. The guiding assumption of this study is that conjectures which students come up with can be initiatives for learning how to prove. This preliminary study will show that problem posing is a strategic tool with potential to bridge everyday instruction and the practice of proving so as to teach how to prove more meaningfully and authentically.

Keywords: Reasoning and Proof, Instructional Activities and Practices, Classroom Discourse

Proof and proving have been considered as central in the context of school mathematics with its roles (Knuth, 2002a) which are "inseparable in doing, communicating, and recording mathematics" (Schoenfeld, 1994). In *Principles and Standards for School Mathematics* (NCTM, 2000), authors argue that "Mathematical reasoning and proof offer powerful ways of developing and expressing insights about a wide range of phenomena." (p. 59). However, it yields difficulty for students to understand it and for teachers to teach it (Stylianides, Stylianides, & Weber, 2010). The disagreement of its centrality for *all* in secondary school mathematics exists among in-service teachers (Knuth, 2002a). Even worse, there are not many opportunities available for students to engage in reasoning and proving in textbooks (Bieda et al., 2014; Thompson, Senk, & Johnson, 2005). Thus, to cultivate a context where students are introduced to proving, engage in the practice, and, ultimately, recognize proving as fundamental in learning of mathematics, need to authentically formulate opportunities beyond those available in textbook should be met. That way, with opportunities to engage in proving in a mathematically meaningful way rather than to take part in a mere ritual as spectators—such as reading and understanding proofs given in textbook without formulating or exploring conjectures, both students and teachers can be more fluent in proving. In this report, a-year-long study of problem-posing activity with particular interest in proving and teacher's instructional interventions which foster student's reasoning and developing proof will be analyzed.

Literature Review

Proof and Reasoning in School Mathematics

Proof and reasoning are neither mere content to be learned with chosen topics nor reserved for certain grade levels. NCTM (2000) states "Reasoning and proof should be a *consistent* part of students' mathematical experience in prekindergarten through grade 12. Reasoning mathematically is a habit of mind, and like all habits, it must be developed through consistent use in *many contexts*." (p. 56, italics added). In the context of secondary school mathematics, the only place in which proof is substantially treated is geometry (Knuth, 2002b). The proofs in the subject and do not show the variety of ways of proving (e.g., proof by contrapositive, reductio ad absurdum). As Thompson, Senk, & Johnson (2005) argued "Because many research studies have shown that writing proofs is difficult for students at all levels, it seems to us that students need more opportunities to engage in varied aspects of proof-related reasoning in order to become more fluent in reasoning and proving."(p. 286) Furthermore, there are not many opportunities for students to engage in reasoning and proving tasks across textbooks which are considered to be primary sources of teaching and learning mathematics (Bieda et al., 2014; Thompson, Senk, & Johnson, 2012). Mathematics teachers need to take on an active role in teaching reasoning and proving beyond what is available in textbook and have strategic knowledge of instructional practice and its relation to student's learning of proof (Stylianides & Ball, 2008) and how it can be more impactful.

In *Proofs and Refutations*, Lakatos (1976) exemplified use of examples when exploring, formulating, qualifying a conjecture, developing a proof and making revisions when encountered with counter examples—either *global* or *local*. Although there exists heterogeneity in appearance among them, mathematically similar objects enable observers to notice regularity between them and the regularity becomes a mathematical conjecture—possibly to be proven true thus to be a theorem. For teacher's specific interest and intent to teach certain theorems, some may argue that designed examples can be given to students as resources to experience transition from empirical arguments to formal proofs. However, the main focus of this study is not on teacher's designing or displaying examples as intended for teaching specific content but on teaching how to strategically generate examples with same constraints in order for students to look for examples (or counter examples) not restricted to those within their reach.

What is problem posing? Silver (1994) defines the term as "both the generation of new problems and the re-formulation, of given problems." (p. 19). According to the author, problem posing can also offer insight into solution of a problem: when developing a proof, posing problems can be a pathway to gain insight into proof. As a way of posing problems, Brown & Walter (1983) suggested "What-If-not?" strategy which new problems can be generated by varying some of the given conditions of a problem. For example, after solving a problem that a sum of two even numbers is even or odd, one can pose a new problem with a question "what would it be if I add two odd numbers?". Lockwood et al. (2013) studied how a mathematician uses examples when proving and disproving. By referring back and forth to examples of relevance to a conjecture, the mathematician gained insight of proof by leveraging idea of one insightful example. In the same line with what Balacheff (1988) called a *generic example*, a representative example of the domain of a conjecture suffices to be developed to a proof by *syntactic proof production* (Weber & Alcock, 2004) or *transformation of images* (Harel & Sowder, 1998). However, unlike teachers and mathematicians, this may be improbable for students to do as such.

Methods

Participants

Geographically located at the vertical center of the Korean peninsula, the school where was the locus of this study is a high school with male students only and located in an urban area. Nearly all students intended to enroll the school to prepare for their admission to college.

As a high school mathematics teacher and the researcher in this project, I had taught junior high students for 3 years and started to teach high school students for the first time by the time this study began. The guiding assumption was that every student is *a theory builder* (Carey, 1985) who can come up with a conjecture or a plausible argument which makes the most sense to them based on their observations and that most of students are able to develop and write proofs by themselves or with a little help offered by *a teacher* or *a more capable peer* (Vygotsky, 1978).

Data Collection

The data collected for the study include student's written assignments, teacher's verbal and written communication with individual students, and two video-taped lessons of which duration is roughly 50 minutes. Based on "What-If-Not?" (Walter & Brown, 1983), for consistency in structure and organization of the assignment, it was structured in a worksheet. Before administering the work sheet weekly, the instructor demonstrated how to use it and explained what is expected as the end result in each step. Until everyone reached understanding of the activity, there had been discussions and negotiations of what it means to be *true*, *valid*, and *appropriate* (Stylianides, 2007; Stylianides et al., 2016) when evaluating validity of a proof.

Data Analysis

Since it may be premature to present a framework which will be used in the later analysis, I shall present the working framework in the process of conducting an initial analysis through the general inductive approach (Thomas, 2006) to highlight themes of relevance to the purpose of this study.

Preliminary Results and Discussions

"What-If-Not" strategy has a potential to offer a strategic way for students to better identify and understand what assumptions are given and conclusions they should prove. For example, one of students in the class was attempting to solve a problem: find the maximum area of a rectangle inscribed in a given isosceles triangle. The student reached at a solution which was not the solution of the problem since he solved a problem without taking the condition "isosceles" into account. Then, after some conversation with him, it came into his attention that he left the condition out. As described in this instance, the student was able to take all the conditions into account after discourse with the teacher. Then, the teacher posed a question as an extension of the problem: "what if the triangle is a right triangle? Or what if the triangle is an acute triangle?" Even though it took a few days for the student to figure out how to solve it, the student reported the teacher that this extended discourse with him led the student to use the strategy in evaluating his understanding of problems by manipulating the given.

Problem posing can offer a strategic way for students to generate examples beyond the individual potential example space (Watson & Mason, 2005) and gain insight into how proof looks like. As it enhances student's understanding of what constraints are given and should be verified by making negation or eliminating and reinstating some of the given and the to-be-verified, it can scaffold student's generation of examples under the conditions met by examples or counter examples. There was a student had issues with qualifying examples and non-examples based on the given constraints. He often said that it is difficult to come up with a counterexample when attempting to disprove a conjecture. The teacher asked the student to think of examples meeting the *least* (i.e. the maximum number of the given conditions as many as he can consider simultaneously into account) subset of the given. Then the teacher demonstrated how to add the rest of the given one by one and accordingly prompted the student to list a number of examples each time. The teacher demonstrated how to qualify examples by adding an additional constraint and left the student doing the rest to reach at being able to generate examples or counterexamples thereafter. The student soon became capable with manipulating the constraints by leaving out and reinstating some of them. This student seemed to show the potential of "what-if-not" strategy as a way of generating examples (or counterexamples) beyond his reach in that he did not try to recall the examples from his experience but qualify examples by adding the given condition into his consideration and narrowing them down to the domain of the argument of his interest. This is not meant to argue that the strategy itself suffices to extend the example space but that it has potential to do as such only with teacher's careful consideration and helpful prompts rather than simply offering the caveat. The kinds of prompts which are crucial in teaching and learning of proof will be identified and discussed in what follows next.

Teacher's role is crucial and critical in the success of teaching and learning of proof. As documented in Stylianou & Blanton (2011), teacher's role becomes of more importance in the teaching proof. In this study, it was the teacher who extended the discourse with individual student to offer an opportunity to engage in exploring and revising conjectures, developing a proof, and prompting student to revise the proof for the greater proximity to the degree of formal proofs. This study will identify three types of prompts by what the teacher intended to elicit from students at a given time: those for *justification, elaboration,* and *generalization*. The instructional intent of prompts was to extend and structure discussions and to attend to what can improve student's proof in terms of precision (in use of mathematical terms, expressions, or representations), clarity (in use of language), and generality (of the proof). The working definitions of the prompts are as follows:

1. *prompts for justification* are meant to point out unexplained parts and request to fill logical gaps or challenge truth of conjectures assumed to be true or referred to in student's argument;
2. *prompts for elaboration* are meant to call attention to what requires clarification or discrepancy between what is intended by student and understood by others; and
3. *prompts for generalization* are meant to pose questions which possibly lead to generalization of part of student's reasoning or examples.

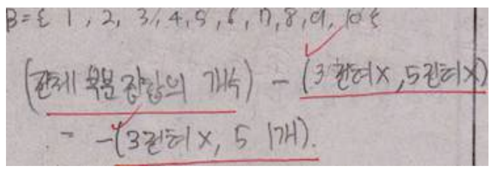

Figure 1. Examples of *Elaboration*

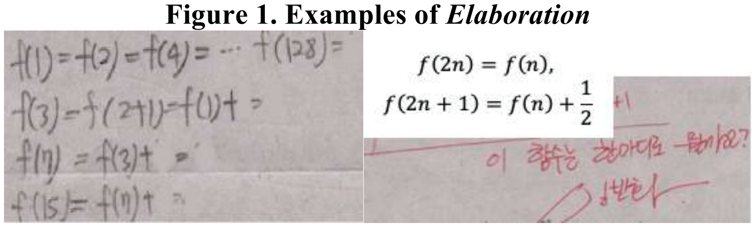

Figure 2. Examples of *Generalization*

There are a few limitations in this study that should be examined through research involving different individual participants, classroom culture, and society. As Cobb & Yackel (1996) pointed out, the results of a well-designed (or well-controlled) research can hardly argue that the study is conducted independently of any aspect of the context of the socio-cultural or individual (or psychological) peculiarity of the participants, the classroom, and the society involved. I acknowledge that this study is not the case that the results are drawn independently of the individuals, the classroom culture, and the society. Future research in the similar perspective toward problem posing and instruction of proof taken in this study will let light be on the unpaved paths I have not taken in this study and nourish the literature.

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