ATTEND TO STRUCTURE AND THE DEVELOPMENT OF MATHEMATICAL GENERALIZATIONS IN A DYNAMIC GEOMETRY ENVIRONMENT

Xiangquan Yao
xzy73@psu.edu
The Pennsylvania State University

Central to mathematical generalization is the development of structural thinking. By examining the relationship between structural thinking and mathematical generalization, this study found that learners' attention to different elements of a problem can result in different mathematical generalizations and structural generalization occurs only when learners reason based on identified properties. These findings imply that learners should be cultivated to attend to mathematical structures and to generalize beyond numerical patterns.

Keywords: Advanced mathematical thinking

Generalizing involves transportation of a mathematical relation from a given set to a new set for which the original set is a subset, perhaps adjusting the relation to accommodate the larger set. It has been argued that generalizing should be at the heart of mathematics activity in school (e.g., Mason, Johnston-Wilder, &Graham, 2005). Within the past a few decades researchers have differentiated different forms of mathematical generalizations (Dörfler, 1991; Yerushalmy,1993; Mason, Burton, & Stacey, 2010), among which are *empirical* and *structural* generalizations. Empirical generalization is the process of forming a conjecture about what might be true from numerous instances. It occurs when a learner looks at several, sometimes many, cases or instances and identifies the sameness among these cases as a general property. Structural generalization arises when a learner recognizes a relationship from one or very few cases by attending to the underlying structure within these cases and perceives this relationship as a general property. The distinction implies the need for learners to move from empirical to structural generalization. Central to this advancement is the development of structure thinking, which can be defined as a disposition to use, explicate, and connect mathematical properties in one's mathematical thinking (Mason, Stephens, & Watson, 2009). However, most studies on generalizing were conducted in the context of pattern recognition. More importantly, by providing the first few terms of a pattern, the tasks used in these studies tend to promote generalization that does not necessarily demand structural thinking (Küchemann, 2010). To extend the study of mathematical generalization beyond the context of pattern recognition and to bring structure thinking to the forefront of the development of mathematical generalization, this study aimed to examine the relationship between structure thinking and mathematical generalization in a dynamic geometry environment (DGE). It was guided by one research question: How does learners' structure thinking evolve and influence their generalizing activity when working on a carefully designed sequence of tasks in DGE?

Theoretical Framework

Mason et al. (2009) described mathematical structure as the identification of general properties that are instantiated in a particular situation as relationships between elements and differentiated five states of learner's attention to mathematical structure. *Holding wholes* involves a certain way of looking at a whole situation that produces a global image that will undergo further analysis. In this awareness state, a learner attends to an object as a whole without explicit regard to its components. *Discerning details* shift the learner's attention toward further analysis and deep description, in which parts begin to be discerned and described in detail based on what the learner finds meaningful to inspect. The attention can focus on parts that either change or remain invariant. *Recognizing* *relationships* occurs when changing or invariant relationships are detected and analyzed critically. In this awareness state, the learner attends to relationships between parts or between part and whole. *Perceiving properties* occurs when the learner perceives the discerned relationships as instantiations of general properties which can apply in many different situations. It involves the transition from seeing something in its particularity to seeing it as representative of a general class. This state enables a further categorization of different (classes of) objects. The separation of stages three and four indicates a subtle but vital difference between recognizing relationships in particular situations and perceiving relationships as instantiations of general properties which can apply in many different situations. *Reasoning on the basis of the identified properties* is the critical phase in which inductive and abductive reasoning about specific objects transforms into deductive reasoning by examining what other objects may belong to the perceived structure. In this awareness state, the learner attends to properties as abstracted from and independent of any particular objects and forms axioms from which deductions can be made. This model provides a useful tool to examine the development of structural thinking.

Methodology

The data for this study was collected from a series of task-based interviews that were a part of a larger research project aimed to investigate preservice secondary mathematics teachers as learners and teachers of mathematical generalizations in a technology-intensive learning environment. The task-based interview was chosen to gain knowledge about individual preservice teacher's processes to generalize mathematical ideas and the mathematical knowledge resulting from it. Each task in this study consisted of a sequence of closely related problems that aimed to promote learners to generalize a mathematical idea to a broader domain. These tasks were design to engage learners in not only empirical but also structural generalizations.

The participants were 8 undergraduate preservice secondary mathematics teachers enrolled in a course that focused on teaching mathematics with various types of mathematical action technologies. The course took a problem-solving approach and engaged the preservice teachers in the processes of representing, conjecturing, generalizing, and justifying by solving and extending mathematically rich problems in technology-rich learning environments. Outside the class each participant participated in four task-based interviews, each of which was about 2 hours. During each interview, a participant would solve one or two mathematical tasks with the technologies they had learned in class. Participants' interactions with technology were screen-recorded. During each session, the interviewer frequently asked the participant to articulate his/her thinking process and to make general statement based on his exploration. Those interactions between the interviewer and the participant were recorded with a camera.

Data analysis consisted of three stages. First, the generalizations a participant constructed while solving each mathematical task were identified and categorized into empirical and theoretical generalizations. A generalization was coded as empirical if it was constructed on the basis of perception or numerical pattern by comparing numerous instances; it was coded as structural if it was constructed based on the generality of the inferred ideas, methods, or processes. Second, Mason et al. (2009)'s model was used to analyzed a participant's evolution of the state of attention to mathematical structure when constructing each mathematical generalization. The final stage involved coordinating the analysis in the first two stages to look for patterns about the evolution of structural thinking and the development of generalization.

Results and Discussion

Results from data analysis indicated a close relationship between the state of attention to mathematical structure and the forms of mathematical generalization that can potentially emerge.

More specifically, the study found that (1) learners' attention to different elements of a problem can result in different mathematical generalizations and (2) structural generalization occurs only when learners can reason on the basis of identified properties. I will use participants' work on task to illustrate the findings from this study. In the task, participants were asked to decide the conditions under which the area of the square created from the largest side of a triangle is equal to the sum of the areas of the squares created from the other two sides of the triangle (Part 1) and to further extend this relationship to quadrilateral (Part 2) and other polygons (Part 3).

When solving Part 1 of the task, Although Joe quickly connected it with the Pythagorean theorem, he focused his attention on the relationship of square from a right leg and the square from the hypotenuse, conjectured that the areas of the two squares grew proportionally and the vertex A shared by the two squares moved along a line, and then validated his conjecture by perception and measurement (Figure 1a). When solving Part 2 of the task, Joe made one interior angle of the quadrilateral a right angle by dragging and then dragged the vertex opposite to the right angle such that the area of the largest square was equal to the sum of the areas of the other three squares. After creating multiple instances of the desired diagram through dragging, informed by his knowledge gained from earlier exploration, he conjectured that the vertex D opposite to the right interior angle moved along a line and the areas of the two squares that share the vertex A grew proportionally (Figure 1b). Here, Joe attended to the relationship between the areas of the two squares and generalized this relationship from triangle to quadrilateral.

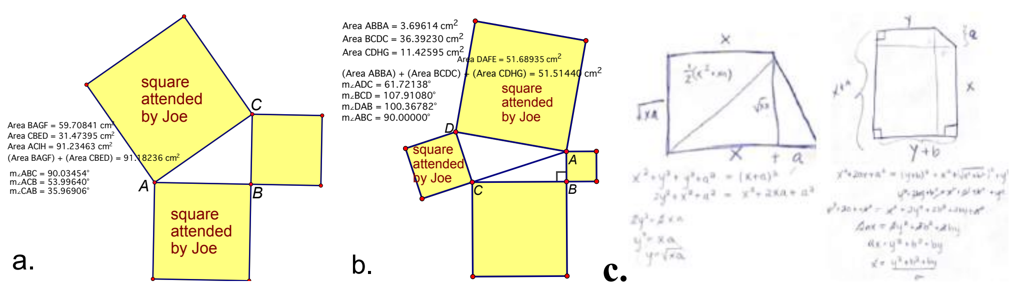

Figure 1: Snapshots of participants' work

In contrast, when exploring Part 2 of the task, Jen considered a right isosceles trapezoid, made two right triangles inside the trapezoid, and labeled the shorter base as x, the longer base as $x + a$, and a lateral side as y (see Figure 1c). By using the fact that the area of the largest square should be equal to the sum of the areas of the three squares and the Pythagorean theorem, she created an equation $x^2 + y^2 + y^2 + a^2 = (x + a)^2$ and concluded that $y = \sqrt{ax}$ after symbolic manipulations. When asked how to further extend the relationship to other polygons, Jen drew a pentagon with three right interior angles as shown in Figure 3 and labeled $x, y, x + a, y + b$ as the length of its four sides. By using the fact that the area of the largest square should be equal to the sum of the areas of the four squares and the Pythagorean theorem, he created an equation $x^2 + 2ax + a^2 = (y + b)^2 + x^2 + (\sqrt{a^2 + b^2})^2 + y^2$ and concluded that $x = \frac{y^2+b^2+by}{a}$ after symbolic manipulation. Moreover, Jen noticed that there was one right angle in the case of triangle, two right angles in the quadrilateral, three right angles in the pentagon and concluded that there would be $n - 2$ right angles by extending the perceived numerical pattern. Here, Jen attended to the desired symbolic relationship between the sides of a polygon and the algebraic identity expressed in the Pythagorean theorem to search for a class of polygons that would satisfy the problem condition. What was generalized was a symbolic relationship rather than the underlying structure expressed in the Pythagorean theorem.

Different from both Joe and Jen, Jack made generalizations by attending the underlying structure of the Pythagorean theorem. The following excerpt shows his generalization of the Pythagorean theorem when solving Part 2 and Part 3 of the problem.

> Interviewer: Now let's think a little bit of what we have done here. What if it is a nonagon, decagon, or an n-sided polygon, how can you create the polygon such that the area of the largest area is equal to the sum of the areas of the other squares drawing from each side of the polygon?
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> Jack: From one of the vertices of the octagon, the vertex on the largest square, I need the side of each square and the line connecting A to each vertex of the nonagon or the n-gon to form a 90-degree angle. So, you need to make $n - 2$ right angles because the only ones that aren't are the two vertices from the largest square.

The above excerpt provides evidence that Jack extended the Pythagorean theorem to any polygon and generalized that the area of the largest square is equal to the sum of the areas of the n-1 squares created from each side of an n-sided polygon when the polygon is created by sequentially drawing n-2 right angles from a vertex of the polygon to the sides of the polygon.

This study found a close relationship between the elements that the participants attended to and the possible mathematical generalizations they might develop. As shown in the above examples, when solving the task, Joe focused on the covariation of the areas of the two squares, Jen attended to the algebraic identity expressed in the Pythagorean theorem, and Jack focused on the structure underlying the Pythagorean theorem. As a result, Joe generalized the proportionality of the areas of the two squares from triangle to quadrilateral, Jen applied the algebraic identity to deduce algebraic equations that specify a given set of quadrilaterals and pentagons that satisfy the problem condition, and Jack extended Pythagorean theorem and used it to decide the particular shape of an n-sided polygon that satisfies the problem condition. One productive way of helping learners to identify mathematical useful relations is to engage them to examine the generalizability of the perceived mathematics relations and the structures behind them.

Pattern generalization is a typical generalization activity in school mathematics, in which a figurative, numerical, or tabular pattern is usually presented in the form a systematic sequence of elements, and learners are expected to generate a systematic set of ordered pairs from which an empirical relationship can be induced. This approach allows learners to identify and express a numerical relationship without necessarily seeing the mathematical structure that produces it. This study found that although the inductive nature of the dynamic geometry environment made it relatively easy for the participants to observe, conjecture, validate, and generalize mathematical relations based on perception and numerical patterns, identifying structure underlying these relations and generalizing them to broader contexts proved to be challenging. For instance, when solving the above task, the participants produced various generalizations relying on measurement and dragging, but only two of them were able to generalize the Pythagorean theorem from triangle to other polygons. A similar result was found in other tasks given to the participants. Therefore, engagement in pattern generalization does not necessarily support learners' development of structural thinking. One plausible reason that many participants in this study were not able to generalize on the basis of mathematics structure is that they were not provided sufficient opportunities to engage in this way of thinking in their own mathematics learning experience. In order to develop learners' ability to make structural generalization, they should be provided opportunities to initiate into structural thinking.

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