A QUANTITATIVE REASONING STUDY OF STUDENT-REPORTED DIFFICULTIES WHEN SOLVING RELATED RATES PROBLEMS

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This paper extends work in the area of quantitative reasoning at the undergraduate level. Task-based interviews were used to examine 16 calculus students' difficulties when solving three related rates problems. Analysis of students' verbal responses and written work revealed several difficulties, including dealing with several time-dependent quantities. The paper concludes with a recommendation for the teaching of related rates problems at the undergraduate level.

Keywords: quantitative reasoning, related rates problems, derivatives, problem solving.

Related rates problems involve at least two rate quantities (i.e., instantaneous rates of change) that can be related algebraically by an equation, function, or formula. Although related rates problems constitute an essential part of any first-semester calculus course in the United States, several researchers have argued that there is a shortage of research that has examined students' thinking about related rates problems at the undergraduate level (e.g., Engelke, 2007; Mkhatshwa, 2020; Speer & King, 2016). Of the few studies involving related rates problems, Mkhatshwa (2020) reported on students who exhibited poor calculational knowledge of the product and quotient rules of differentiation, something that limited their success in a non-routine related rates problem they were asked to solve. Engelke (2007) described beneficial components of a successful solution to a related rates problem, including drawing a diagram, determining a functional relationship (algebraic equation), and checking the answer for reasonability. Other studies have found that mathematizing (Freudenthal, 1993) related rates problems is problematic for students (Martin, 2000; White & Mitchelmore, 1996).

While these studies have provided useful information about how students set up and solve related rates problems, there is still much to be explored about what different modes of reasoning, such as quantitative reasoning (Thompson, 1993, 1994b, 2011) might reveal about students' difficulties with solving related rates problems that have real-world contexts such as kinematics. Quantitative reasoning seems a particularly important lens for studying students' understanding of related rates problems since they inherently deal with quantities. In addition, students' difficulties with solving related rates problems in these studies have all been reported from a researcher's perspective (i.e., observed difficulties), and have not considered a student perspective (i.e., student-reported difficulties). Thus, in order to build on these studies, the present study investigated students' difficulties with solving related rates problems from a student perspective. The research question we investigated is: What do calculus students identify as difficulties when engaged in reasoning quantitatively about solving related rates problems?

Related Literature

Evidence from studies that have examined students' reasoning about geometric related rates problems (Mkhatshwa, 2020) shows that students who are able to visualize and perform physical enactments of situations described in related rates problems tend to be successful in solving these problems (Carlson, 1998; Carlson, Jacobs, Coe, Larson, & Hsu, 2002; Monk, 1992). Several researchers have identified lack of facility with implicit differentiation as a major cause for students' failure to solve related rates problems successfully (Clark et al., 1997; Engelke, 2004; Mkhatshwa, 2020; Piccolo & Code, 2013). Piccolo and Code (2013) argued that students' difficulties with solving

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related rates problems stem from a weak understanding of implicit differentiation, rather than a misunderstanding of the physical context of such problems. Hare and Phillippy (2004) posited that "implicit differentiation is a difficult concept for many students to understand because the level of difficulty of the concept is higher than the level of difficulty of explicit functions" (p. 7). Conflicting findings have been reported on calculus students' ability to mathematize related rates problems (cf., Martin, 2000; Mkhatshwa, 2020; White & Mitchelmore, 1996). Analysis of students' written responses to geometric related rates problems by Martin (2000) revealed that overall performance was poor, and that "the poorest performance was on steps linked to conceptual understanding, specifically steps involving the translation of prose to geometric and symbolic representations" (p. 74). Findings of a recent study (Mkhatshwa, 2020) on students' thinking about related rates problems in real-world contexts indicated that mathematizing routine related rates problems is straightforward for students.

Theoretical Perspective

This study draws on the theory of quantitative reasoning (Thompson, 1993, 1994b, 2011). Quantitative reasoning is the act of analyzing a problem in terms of the quantities and relationships between the quantities involved in the problem (Thompson, 1993). In this study, quantitative reasoning refers to how students interpreted rate quantities (i.e., instantaneous rates of change) when solving related rates problems, and how they reasoned about quantities and relationships between quantities when engaged in talking about difficulties they had with solving these problems. What is important in quantitative reasoning is making sense of quantities and relationships between quantities (Smith III & Thompson, 2007; Thompson, 1993). Thompson (2011) described three tenets that are central to the theory of quantitative reasoning, namely a quantity, a quantitative operation, and quantification. A quantity is a measurable attribute of an object (Thompson, 1994b). Examples of quantities in this study include the speed of an airplane, the area of a puddle, and the volume of a balloon. A quantitative operation is the process of forming a new quantity from other quantities (Thompson, 1994b). We designed three tasks (Task 1, Task 2, and Task 3 in the methods section) that provided opportunities for students to perform quantitative operations by creating new quantities through the process of implicit differentiation. Quantification is the process of assigning numerical values to quantities (Thompson, 1994b). The three tasks used in this study provided opportunities for students to engage in quantification.

Methods

Task-based interviews (Goldin, 2000) were used to investigate calculus students' quantitative reasoning while solving related rates problems. The interviews covered three tasks:

> **Task 1 [motion context]:** Two small planes approach an airport, one flying due west at a speed of 100 miles per hour and the other flying due north at a speed of 120 miles per hour. Assuming they fly at the same constant elevation, how fast is the distance between the planes changing when the westbound plane is 180 miles from the airport and the northbound plane is 200 miles from the airport?
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> **Task 2 [non-motion context]:** A leak from the sink is creating a puddle that can be approximated by a circle, which is increasing at a rate of $12\ cm^2$ per second. How fast is the radius growing at the instant when the radius of the puddle equals 8 cm?
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> **Task 3 [non-motion context]:** For the next problem, let me give you a little background on a formula that we will use. Suppose a gas is inside a container. Many gases under normal conditions follow the "ideal gas law," $PV = kT$, where P is the pressure the gas exerts on the container, V is the volume of the container, T is the temperature of the gas, and k is a constant. P is measured in "atmospheres," V is measured in cubic meters, and T is measured

> in Kelvins. Kelvins is a lot like Celsius, except that it is scaled so that 0 means absolute zero (lowest possible temperature), which makes water's freezing point to be $273\ °K$. Do you have any question(s) about this formula, or any of the quantities [like temperature in Kelvins] before we proceed?
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> In a laboratory, an experiment is being done on a gas inside a large flexible rubber balloon. For the experiment, the temperature of the gas is being heated at a rate of 8 degrees Kelvin per second. At one point, when the temperature of the gas is $300\ °K$, the pressure is 1.5 atmospheres, the volume of the gas is one cubic meter, and the volume of the gas is increasing at a rate of 0.01 m^3 per second. At that moment, is the pressure in the balloon increasing or decreasing? What is the rate of that increase/decrease?

After students concluded their work on each task, the interviewer asked the following questions: (i) What does your answer [derivative] tell you in the context of this task? (ii) How would you answer the question posed in this task? (iii) What was the easy part for you when solving this task? and (iv) What was the difficult part for you when solving this task? With these questions, our goal was two-fold. First, we wanted to examine students' interpretations of derivatives in motion and non-motion contexts (questions i. and ii.). Second, we wanted to gain an insight on what is straightforward and what is difficult about solving related rates problems from a student perspective (questions iii. and iv.).

Setting, Participants, Data Collection, and Data Analysis

The study participants were 16 undergraduate students at a research university who were enrolled in five different sections of a calculus I course taught by three different professors. Details about the participants, including opportunities they had to learn about related rates problems during classroom instruction are provided in Mkhatshwa (2020). Data for the study consisted of transcriptions of video-recordings of the task-based interviews and work written by the students during each interview session. On average, each interview session lasted for about 65 minutes. The data was analyzed in two stages. In the first stage, we used two emergent codes i.e., student actions that evolved from the data. These codes are: (1) the difficulty of dealing with several time-dependent variables (quantities), and (2) the difficulty of finding the value of the constant k in Task 3. In the second stage of the analysis, we tallied the number of students in each of codes found in the first stage of the analysis.

Results, Discussion, and Conclusions

Since Task 3 was the only non-routine task, and one that most of the students were least successful in solving, we limit our discussion of student-reported difficulties with solving related rates problems to this task. There are three findings from this study. First, when asked about the difficult part about solving the task, six students stated that everything about the task was hard. Amos' reasoning about the difficulty of solving Task 3 is representative of the six students.

Researcher: What was the easy part for you when solving this task [Task 3]?
Amos: None of the problem was easy for me.
Researcher: What was the difficult part for you when solving this task?
Amos: Reading the task, differentiating the given equation [$PV = kT$], and figuring out where to plug in the given values [quantities in the task, e.g., the temperature of the gas given as $300°K$] in order to solve the problem.

In response to the first question (i.e., the easy part), Amos stated that "none of the problem was easy" for him. When asked about the difficult part, he noted reading the task, differentiating the equation given in the task, and using all the given information in the task to solve the problem posed in the task. When probed about the type of differentiation he would use in this task, Amos stated that he would "have to use implicit differentiation." Two other students identified implicit differentiation as the challenging part when solving the task. Students' difficulties with implicit differentiation (i.e., performing quantitative operations) when solving related rates problems have been reported by other researchers (cf., Clark et al., 1997; Mkhatshwa, 2020; Piccolo & Code, 2013).

Second, four other students stated that there were several variables (quantities) to keep track of, and that this was the main challenge for them when solving the task. The following excerpt illustrates how Felix, whose reasoning is representative of these students, commented about the difficult part when solving the task.

Researcher: What was the difficult part for you when solving this task?
Felix: There are more than two variables [quantities], P, V, and T in the same equation [$PV = kT$].

Felix remarked that having several quantities, namely pressure (P), volume (V), and temperature (T) in the same problem was problematic for him when solving the problem posed in the task. He, however, did not elaborate on this. We argue that the unfamiliar context may have been the challenge for Felix more than having several variables. This is because in Task 1 (a familiar task to Felix and one that has several variables as well), Felix did not claim that having several variables in the problem was the difficult part. Instead, he said the difficult part was finding an equation that relates the quantities in the task, that is, mathematizing the problem. On the contrary, 10 students claimed that Task 2 (a routine task) was easy to solve because it had fewer variables compared to Task 1 and Task 3. When asked about the easiest part about solving Task 2, one of the 10 students, James, commented, "we only had one variable to track and that's the radius, so it was fairly easy to solve." When asked about the difficult part when solving Task 2, he said, "I don't think there were any challenges." We argue that the number of variables play a huge role in students' ability to solve related rates problems successfully.

Third, four students stated that finding the value of the constant k was the difficult part for them when solving Task 3. We note that although only four students identified solving for the constant k as the problematic part in Task 3, half of the 16 students in this study were unsuccessful in finding the value of k. Since finding the value of the constant k entails engaging in the process of quantification i.e., substituting the given values of the quantities of P, V, and T in the equation $PV = kT$ and then solving for k, we argue that substituting known quantities and solving for an unknown quantity in an equation is perhaps not only problematic for secondary school students, but also for undergraduate students. Based on the student-reported difficulties when solving related rates problems in this study, we recommend that calculus instruction should provide more opportunities for students to make sense of, and to solve non-routine related rates problems that have several quantities. The interested reader is referred to Mkhatshwa (2020) for observed (i.e., researcher-reported) difficulties that were exhibited by the students when solving the three tasks used in the present study.

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