BEYOND PATTERNS: MAKING SENSE OF PATTERNS-BASED GENERALIZATIONS THROUGH EMPIRICAL RE-CONCEPTUALIZATION

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Identifying patterns is an important part of mathematical reasoning, but many students struggle to justify pattern-based generalizations. Some researchers argue for a de-emphasis on patterning activities, but empirical investigation has also been shown to support discovery and insight into problem structures. We introduce a phenomenon we call empirical re-conceptualization, which is the development of a generalization based on an empirical pattern that is subsequently re-interpreted from a structural perspective. We define and elaborate empirical re-conceptualization by drawing on data from secondary and undergraduate students, and identify three major affordances: Empirical re-conceptualization can serve as (a) a source of verification, (b) a means of justification, and (c) a vehicle for generating insight.

Keywords: Reasoning and Proof, Cognition, Algebra and Algebraic Thinking

Objective: Leveraging the Power of Pattern-Based Generalizations

Recognizing and developing patterns is a critical aspect of mathematical reasoning. Many students are adept at recognizing and formalizing patterns (Pytlak, 2014), but they can also struggle to understand, explain, and justify those very patterns they develop (Čadež & Kolar, 2014). One source of students' difficulties may rest with the empirical nature of those generalizations. Students can become overly reliant on examples and infer that a universal statement is true based on a few confirming cases (Knuth, Choppin, & Bieda, 2009). One potential solution is to help students understand the limitations of empirical evidence and thus recognize the need for deductive arguments (e.g., Stylianides & Stylianides, 2009). These approaches have shown some success in helping students see the limitations of examples, but they also frame empirical reasoning strategies as stumbling blocks to overcome.

In contrast, we have identified a phenomenon that we call *empirical re-conceptualization*, in which students identify a pattern, form an associated generalization, and then re-interpret their findings structurally. From this perspective, students can bootstrap their pattern-based generalizations into mathematically meaningful insights and arguments. In this paper, we describe and elaborate the construct of empirical re-conceptualization and address the following questions: (a) What characterizes students' abilities to leverage pattern-based generalizations in order to develop mathematical insights? (b) What are the conceptual affordances of empirical re-conceptualization? We offer a secondary example, discuss the affordances experienced, and consider ways in which instruction can support the practice of empirical re-conceptualization.

The Drawbacks and Opportunities of Empirical Reasoning

While an emphasis on patterning that lacks meaning can promote the learning of routine procedures without understanding (Fou-Lai Lin et al., 2004), there are also a number of affordances that can arise from empirical investigation. The act of developing empirically-based generalizations can foster the discovery of insight into a problem's structure, which could consequently support proof development (de Villiers, 2010). The degree to which pattern generalization is an effective route to proof is an open question, but there is evidence that students can and do engage in a dynamic interplay between empirical patterning and deductive argumentation (e.g., Schoenfeld, 1986).

In: Sacristán, A.I., Cortés-Zavala, J.C. & Ruiz-Arias, P.M. (Eds.). (2020). *Mathematics Education Across Cultures: Proceedings of the 42nd Meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education*, Mexico. Cinvestav / AMIUTEM / PME-NA. https:/doi.org/10.51272/pmena.42.2020

Students lack sufficient experience with developing meaning from patterns. Curricular materials emphasize patterning activities that end with a generalization, typically an algebraic rule; developing an associated justification is seldom emphasized in standard classroom tasks. In fact, students typically receive little, if any, explicit instruction on how to strategically analyze examples in developing, exploring, and proving generalizations (Cooper et al., 2011). We propose that empirical re-conceptualization can be one way to provide opportunities to develop mathematical insight and deductive argumentation from pattern-based generalizing activities.

Theoretical Perspectives: Structural Reasoning

Harel and Soto (2017) identified five major categories of structural reasoning: (a) pattern generalization, (b) reduction of an unfamiliar structure into a familiar one, (c) recognizing and operating with structure in thought, (d) epistemological justification, and (e) reasoning in terms of general structures. The first category further distinguishes between result pattern generalization (RPG) and process pattern generalization (PPG) (Harel, 2001). RPG is a way of thinking in which one attends solely to regularities in the result. The example Harel gave is observing that 2 is an upper bound for the sequence $\sqrt{2}$, $\sqrt{2+\sqrt{2}}$, $\sqrt{2+\sqrt{2+\sqrt{2}}}$, … because the value checks for the first several terms. When we refer to empirical re-conceptualization and the identification of a pattern based on empirical evidence, we are referring to RPG. In contrast, PPG entails attending to regularity in the process. Harel discussed how one might engage in PPG to determine that there is an invariant relationship between any two consecutive terms of the sequence, $a_{n+1} = \sqrt{a_n + 2}$, and therefore reason that all of the terms of the sequence are bounded by 2 because $\sqrt{2} < 2$.

We define empirical re-conceptualization as the process of re-interpreting a generalization based on RPG from a structural perspective. By structural perspective, we mean engaging in any of the following activities: (a) shifting from RPG to PPG; (b) reducing an unfamiliar structure into a familiar one; (c) carrying out operations in thought without performing calculations; (d) forming and reasoning with a new conceptual entity; or (e) shifting from figurative to operative activity. In short, re-interpreting a generalization from a structural perspective entails the ability to recognize, act upon, and reason with general structures.

Methods

Barney (a 7th-grade student) and Homer (a 9th-grade student) participated in a paired teaching experiment (Steffe & Thompson, 2000), which took place across five sessions averaging 75 minutes each. An aim of the teaching experiment was to investigate the students' generalizations about the areas and volumes of growing figures, and then to study their development of combinatorial reasoning by exploring the growing volumes of hypercubes and other objects in 4 dimensions and beyond.

All teaching sessions were videoed and transcribed. We first drew on Ellis et al.'s (2017) RFE Framework to identify generalizations, and then used open coding to infer categories of generalizing activity based on the participants' talk, gestures, and task responses. We then identified an emergent set of relationships between the participants' patterning activities and the types of generalizations they formed; this yielded the category of empirical re-conceptualization. In a final round we re-visited the data corpus in order to identify all instances of empirical re-conceptualization, the generalizations that led to each instance, and the subsequent explanation or justification. In this manner we were able track the changes in students' activity after engaging in re-conceptualizing, which led to the identification of the affordances detailed below.

Results

We found three major affordances of engagement in empirical re-conceptualization. Namely, empirical re-conceptualization can serve as (1) a source of *verification*, (2) a means of *justification*, and (3) a vehicle for *generating insight*. Within the third category, we identified three types of insight: (3a) re-interpretation within a different context or representational register, (3b) the creation of a new generalization, and (3b) the establishment of a new piece of knowledge. In order to characterize the phenomenon of empirical re-conceptualization and its associated affordances, we present an exemplar case.

Secondary Case: Growing Volumes in Three Dimensions and Beyond

Barney and Homer explored the added volumes of three-dimensional, four-dimensional, and other n-dimensional "cubes" that grew uniform amounts in every direction. They began by determining the added volume of an n by n by n cube that grew 1 cm in height, width, and length. The students worked with physical cubes to consider the component pieces and determined that the added volume would be $3n^2 + 3n + 1$. When they then investigated the added volume of a cube that grew x cm in each direction, the students simply generalized from their prior result. Homer wrote "$(3x)n^2 + (3x)n + x^2$", replacing the 3 in the first two terms of his original expression with a $3x$, and replacing the 1 in the last term, which he had conceived as 1^2, with an x^2. Unsure about the correctness of this expression, Barney said, "let me model on the cube", which he used to verify that the first term, $3xn^2$, was correct because it represented three additional rectangular prisms, each with a volume of xn^2. Both students then realized errors in the next two terms. Barney explained that the second term should actually be $3x^2n$ "because you're adding 3 of x by x by n." Both students also realized the final term would have to be x^3.

The students' original generalization was based on the *result* of their prior activity in building up additional volume components, rather than attending to the *process* by which they grew the cube's volume. However, Barney then experienced a need to verify Homer's result, which led to re-conceptualizing the generalization within the context of volume. He took the algebraic structure and made sense of it geometrically, in the process coordinating his mental activity of constructing component volumes and translating those quantities to algebraic representations.

The students eventually went on to determine expressions of added volume for the 2nd, 3rd, and 4th dimensions, which the teacher-researcher wrote in Figure 1. Homer then saw a pattern in the expressions, exclaiming, "Oh, I know what's happening!":

> Homer: It is simple, as 2 – sorry I'm writing on it. [Begins to draw the blue lines.] Two plus 1 is 3, and 2 plus 1 is 3, 3 plus 3 is 6, 3 plus 1 is 4, 1 plus 3 is 4. [Writes the red numbers.]
> TR: Whoa. Huh.
> Barney: Wow. It's just that one triangle, Pascal's triangle, right?

Homer recognized the pattern in which each coefficient could be determined by adding the sum of the coefficients of the prior consecutive terms. Pascal's triangle then became a mechanism for determining the additional volume of a 5th-dimensional solid, which the students wrote as "$5n^4 + 10n^3 + 10n^2 + 5n^1 + 1^5$". They then decided to check their answer by listing the arrangements of three ns and two 1s (the $10n^3$) case, which served to verify that the coefficient was indeed 10. Barney then realized that given that they had verified the $10n^3$ case, they did not need to check the $10n^2$ case: "We can basically just take this and switch all the ns to 1s and 1s to ns." This explanation of symmetry caused Homer to then extend that finding to new cases: "Oh, and you know what? You can do the same for these (pointing to the $5n^4$ and the $5n^1$ terms)…you can just replace these 1s for ns."

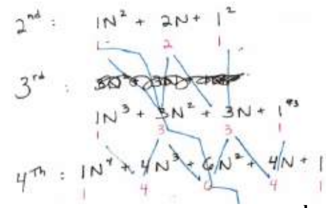

Figure 1: Expressions for added volume in the 2nd, 3rd, and 4th dimensions

Homer and Barney initially developed a generalization based on Pascal's triangle, which allowed them to determine the expression for added volume. Their subsequent listing activity enabled the students to re-interpret that expression combinatorially. That pattern allowed the students to engage in a verification process and subsequently reason about outcomes to develop a new insight, that there must be symmetry in the coefficients. Barney was able to reflect on his operations in listing the ten outcomes and realize that there was nothing special about the characters n and 1, and that they could simply be reversed in the case of determining the combinations of two ns and three 1s. This then supported Homer's new generalization.

Discussion

Empirical re-conceptualization can serve as a source of *verification*, such as when Barney checked the algebraic expression for adding x cm to a cube by appealing to the notion of volume. It can also serve as a source of *justification*, which we saw when Barney justified Homer's pattern of xs in the expression $3xn^2 + 3x^2n + n^3$. We also saw the students developing *insight*. They developed new knowledge and understanding, such as when Barney generated the idea that the coefficient of n^3 must be identical to the coefficient of n^2, which then supported Homer's ability to establish a new generalization that could be extended to the other terms, $5n^4$ and $5n$.

These affordances suggest that empirical re-conceptualization can serve as a vehicle to transform empirical patterns into meaningful sources of verification, justification, and insight. Certainly, students may also identify and generalize patterns that they do not understand or cannot justify. A danger is that students will engage in empirical investigation but then not seek to re-conceive their findings structurally. We find it useful to explore the conditions that can best support students' transition to the productive next step, that of empirical re-conceptualization. Our data suggest that directing students back towards the contextual genesis of the patterns they generalize may be an effective strategy for supporting empirical re-conceptualization. With the support of concrete contexts for meaning making, the activity of generalizing empirical patterns can serve as a bridge to more generative and productive mathematical activity.

Acknowledgments

The research reported in this paper was supported by the National Science Foundation (grant no. DRL-1419973).

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