EXPLORING SHIFTS IN A STUDENT'S GRAPHICAL SHAPE THINKING

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In this report, we present results from semi-structured clinical interviews with a preservice secondary teacher which were conducted prior to and after a teaching experiment intended to support the student in developing emergent graphical thinking. We illustrate how the student engaged in static shape thinking in the pre-clinical interview to describe relationships represented graphically. In the post-clinical interview, the student used both static and emergent reasoning to describe relationships. Hence, we provide an empirical example of a student developing more sophisticated graphing meanings while underscoring the importance of probing students' shape-based thinking.

Keywords: Algebra and Algebraic Thinking, Preservice Teacher Education

Researchers have shown that graphs represent information and relationships in ways that are difficult to express in other forms (e.g., Arcavi, 2003) and provide insights into students' thinking and learning (e.g., Moore et al., 2014). However, students and teachers experience persistent difficulties creating and interpreting graphs (e.g., Clement, 1989; Leinhardt et al., 1990). For instance, students often treat graphs as literal representations of a situation (e.g., interpreting a time-speed graph of a biker as the bikers' traveled path). The research examining students' graphing meanings indicates common instructional approaches do not provide students sustained opportunities to develop meaningful ways of representing relationships between covarying quantities. These failings may stem from the fact that covariational reasoning is generally absent in U.S. school curricula (Thompson & Carlson, 2017). Hence, in this paper we leverage Moore and Thompson's (2015) construct of graphical shape thinking to explore the research question: Can (and if so how can) a student whose meanings for interpreting graphs are constrained to shape-based thinking reorganize her meanings to include emergent thinking?

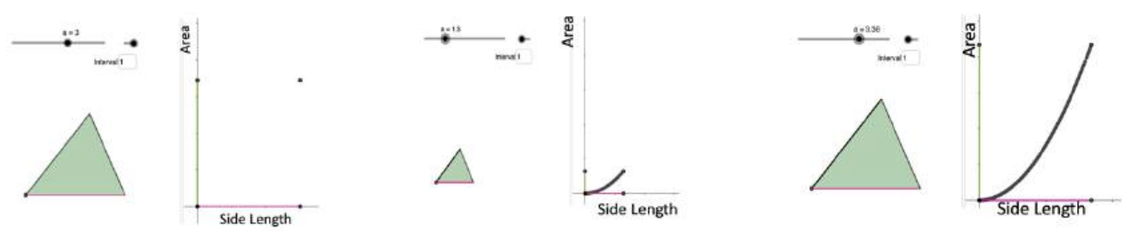
Methods, Participants, and Analysis

This report is situated in a larger teacher experiment (Steffe & Thompson, 2000) that sought to examine two preservice teachers (hereafter students) developing meanings for quadratic and exponential relationships via their covariational reasoning. Here we focus on one student, Josie (pseudonym). Josie was enrolled in a secondary mathematics teacher education program at a large university in the northeast U.S. and had completed a calculus sequence. We present data collected during the clinical interviews prior to and after the teaching experiment to provide insights into Josie's mathematics at the outset of the study and to explore shifts in her meanings at the end of the study. Two members of the research team were present at each interview and each session was video and audio recorded. In order to analyze the data, we used generative and convergent approach (Clement, 2000) in combination with conceptual analysis (Thompson, 2008). With the goal of characterizing Josie's meanings, we used an iterative approach to construct viable models of her meanings and ways of reasoning. During retrospective analysis, we re-watched all interview and teaching sessions to identify instances that provided insights into Josie's static and emergent shape thinking, which we used to develop initial models of Josie's mathematics. We compared these models to researcher notes taken during on-going analysis. When evidence contradicted our initial models, we made new conjectures, including the possibility of shifts in Josie's meanings, and refined our models with the new conjectures in mind. This process resulted in viable characterizations of Josie's mathematics.

Theoretical Perspective: Graphical Shape Thinking

Extending previous characterizations of students' quantitative and covariational reasoning (see Thompson & Carlson, 2017), Moore and colleagues (Moore & Thompson, 2015; Moore, 2016) described two types of *graphical shape thinking* students leverage when constructing or interpreting graphs. Moore and Thompson (2015) characterized static graphical shape thinking as entailing "actions based on perceptual cues and the perceptual shape of a graph" (p. 784). Static shape thinking may include *associations* between the shape of the graph, function name, and analytic rules. For example, a student may associate a parabolic graph (or "U-shape") with the term "quadratic" and a rule of the form "$y = \mathrm{a}x^2 + \mathrm{b}x + \mathrm{c}$". While such associations likely have been productive for a student as she addressed tasks in school mathematics (e.g., shifting 'parent' functions), these associations may not support students when addressing a novel task (e.g., determining a relationship from a data set) or representation (e.g., the polar coordinate system).

Whereas, static shape thinking involves treating a graph as an object, Moore and Thompson (2015) described emergent thinking as a student conceptualizing "a graph *simultaneously* as what is made (a trace) and how it is made (covariation)" (p.784). A student thinking emergently conceives a graph as an in-progress trace representing two covarying quantities magnitudes or values. For example, consider the *Growing Triangle Task* which shows a scalene growing triangle (https://bit.ly/2BjdEKZ). Students are asked to represent the relationship between the triangle's base (in pink) and area (in green). To reason emergently, the student must first construct a coordinate system that represents each quantity on an axis and understand a point in this coordinate system as simultaneously representing both quantities' magnitudes (Figure 4a). The student can then imagine how this point will move in the coordinate system as the triangle's area and side length vary; reasoning emergently entails understanding the graph of the relationship as being produced by the trace of this point as the quantities covary (Figure 1b/c).



(a) (b) (c)

Figure 4. The Growing Triangle Task

Results

In both clinical interviews, we provided Josie two graphs representing quadratic and exponential relationships and asked her to identify the relationship represented by each graph (Figure 5a/b). We hoped to gain insight into Josie's meanings for graphs and her ways of identifying relationships. We also provided tables (e.g., Figure 2c) of values representing quadratic and exponential relationships to explore the connections between Josie's meanings for these relationships in different representations. We note during the teaching episodes, we engaged the students in tasks designed to support their reasoning covariationally and emergently (e.g., the *Ferris Wheel Task* as described by Carlson & Moore, 2012; Moore, 2014). While Josie's activities addressing these tasks was critical to her reorganizing her graphing meanings, we focus on data from the pre- and post-clinical interviews to highlight shifts in her graphical shape thinking for brevity's sake.

Pre-Clinical Interview

When asked to identify the relationship represented by the graph in Figure 5a, Josie claimed:

> It looks like half a parabola, I would say x^2… but I would have to see the other half, definitely to clarify. But it looks like it is going to come back up (*motions as if making a curve to the left of the graph with her hands*)...yeah...It [the graph] doesn't look like, it's, it's… to me if it [the graph] is exponential, they start close to kind of here (*showing the intersection of the axes*), they are closer to zero and shoot up ....so, to me it's [the graph in Figure 2a] half a parabola.

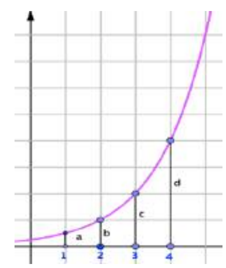
For the graph in Figure 5b, Josie explained, "this one looks like an exponential growth… it started from something very close to zero and then increases very fast. Yeah, that's what I am thinking, it's an exponential growth." We infer Josie leveraged static shape thinking as she determined the relationship represented by each graph. Specifically, she focused on the shape of the graph (e.g. "half a parabola", "started… close to zero and then increases very fast") which she associated with a functional class and analytic rules (e.g. x^2, "exponential growth").

(a)

(b)

x	y
1	3
2	12
3	27
4	48
5	75

(c)



(d)

Figure 5: A graph representing (a) quadratic and (b) exponential relationship, (c) Table of values representing quadratic relationship, (d) a recreation of Josie's explanation

Although Josie's meanings grounded in static shape thinking supported her in correctly describing that Figure 2a and b represented a quadratic and exponential relationship, respectively, her meanings did not entail a way to describe the same types of relationships represented in a table. For example, after determining several consecutive slopes to determine the relationship in Figure 5c was non-linear, Josie noted "All the y's are multiples of three and it is rapidly increasing, maybe it could be an exponential growth." We infer that Josie's meanings for determining a relationship from a table only supported her in determining if a relationship was linear or non-linear; one possible explanation for this is that her meanings for non-linear relationships (e.g., quadratic, exponential) entailed mostly shape-based associations.

Post-Clinical Interview

After having multiple opportunities to construct and graphically represent covariational relationships in the teaching experiment, we engaged Josie in the post-clinical interview 6 weeks after the last teaching episode. Addressing the same problems in Figure 2, she initially engaged in static shape thinking as she relied on visual properties of the curve. However, in this case Josie was able to unpack her thinking via her covariational and emergent reasoning to make claims about the relationships represented by each graph and table. For example, for Figure 5a, Josie first responded, "[the graph] looks quadratic… it is increasing and something like that is either exponential or quadratic or cubic." She further described, "it looks like half a parabola" and making hand motions as if drawing a parabolic curve to the left of the vertical axis explained, "then it would be a parabola and it would be a quadratic." However, Josie then claimed, "let's see if the amounts of amounts of change are the same" and determined two points, (1,2) and (3,8), on the curve, and calculated the differences in the y-coordinates as 6. She said, "if I knew the next one point, I could check if the amounts of amounts of change are same and it would be quadratic" and concluded, "but it looks like a quadratic and I am going to say quadratic." Hence, Josie's meanings now entailed that a graph represents quadratic change if the amounts of change of amounts of change are constant, a defining characteristic of quadratic change.

As a second example of Josie unpacking her static shape thinking, consider her response when determining the relationship represented by the graph in Figure 5b. She first claimed the relationship as exponential because the graph "is starting off slow and then shoots up." After this, she imagined points on the horizontal axis and motioned as if drawing vertical distances from these points to the graph (see Figure 5d for a recreation of her hand motions) and described, "for equal changes in the x's", and motioning her fingers on imaginary segments as seen in Figure 5d "like from 1 to 2 we are not increasing... maybe a half, but 2 to 3 we are increasing maybe by a one, 3 to 4 we are increasing by a two maybe, that is a little more and so on and so forth, and then it increases by more. So, definitely I am going to say it is exponential." In each case, rather than being constrained by making shape-based associations, Josie's meanings for interpreting graphs (and tables, like Figure 5c) now included being able to unpack a graph in terms of the relationship between covarying quantities (e.g., constant second differences for quadratic, increasing by amounts that themselves increase by a factor of two for exponential).

Discussion

In this report, we characterize a student's static and emergent shape thinking before and after a teaching experiment designed to support her in developing meanings for graphs as emergent traces. Addressing our research question, the student was able to reorganize her shape-based meanings for interpreting graphs to meanings that entailed interpreting graphs according to the underlying covariational relationships they represented. We conjecture the numerous opportunities Josie had to reason about and represent relationships between covarying quantities in the teaching experiment supported her in moving beyond static shape thinking.

Second, and addressing the "and if so how" part of the research question, we note how Josie's meanings for interpreting graphs included elements that appeared static in the post-interview. For instance, she continued to use shape-based associations, which is not surprising as these associations can still be useful in determining the relationship represented by a graph. However, as she justified her choice, Josie was able to unpack the graph in terms of the relationship represented by the covarying quantities, which is indicative of her emergent shape thinking. Hence, Josie's activity highlights how a student may still use shape-based meanings while being able to unpack these meanings in terms of the underlying relationship. This underscores the importance of researchers carefully attending to students' graphing meanings. A student engaging in shape-based activity does not mean they are constrained to such reasoning; it is important to examine if the student can unpack their thinking further.

Acknowledgments

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References

Arcavi, A. (2003). The role of visual representations in the learning of mathematics. Educational Studies in Mathematics, 52(3), 215–241.

Carlson, M. P., Jacobs, S., Coe, E., Larsen, S., & Hsu, E. (2002). Applying covariational reasoning while modeling dynamic events: A framework and a study. *Journal for research in mathematics education*, 33(5), 352-378.
Clement, J. (1989). The concept of variation and misconceptions in Cartesian graphing. *Focus on Learning Problems in Mathematics, 11*(1–2), 77–87.
Ellis, A. B. (2011a). Algebra in the middle school: Developing functional relationships through quantitative reasoning. In J. Cai & E. Knuth (Eds.), *Early algebraization* (pp. 215-238): Springer Berlin Heidelberg.
Ellis, A. B. (2011b). Generalizing-promoting actions: How classroom collaborations can support students' mathematical generalizations. *Journal for research in mathematics education*, 42(4), 308-345.
Leinhardt, G., Zaslavsky, O., & Stein, M. K. (1990). Functions, graphs, and graphing: Tasks, learning, and teaching. Review of Educational Research, 60(1), 1–64.
Lobato, J., Hohensee, C., Rhodehamel, B., & Diamond, J. (2012). Using student reasoning to inform the development of conceptual learning goals: The case of quadratic functions. *Mathematical Thinking and Learning*, 14(2), 85-119.
Moore, K. C., Silverman, J., Paoletti, T., & LaForest, K. (2014). Breaking conventions to support quantitative reasoning. *Mathematics Teacher Educator*, 2(2), 141-157.
Moore, K. C. (2016). Graphing as figurative and operative thought. In C. Csíkos, A. Rausch, & J. Szitányi (Vol. Eds.), *Proceedings of the 40th conference of the international group for the psychology of mathematics education*: Vol. 3, (pp. 323–330).
Steffe, L. P., & Thompson, P. W. (2000). Teaching experiment methodology: Underlying principles and essential elements. In R. A. Lesh & A. E. Kelly (Eds.), *Handbook of research design in mathematics and science education* (pp. 267-307). Hillside, NJ: Erlbaum.
Strauss, A. L., & Corbin, J. M. (1998). *Basics of qualitative research: Techniques and procedures for developing grounded theory* (2nd ed.). Thousand Oaks: Sage Publications.
Thompson, P. W. (2008). Conceptual analysis of mathematical ideas: Some spadework at the foundations of mathematics education. In O. Figueras, J. L. Cortina, S. Alatorre, T. Rojano, & A. Sépulveda (Eds.), *Proceedings of the Annual Meeting of the International Group for the Psychology of Mathematics Education* (Vol. 1, pp. 31–49). Morélia, Mexico: PME.
Thompson, P. W., & Carlson, M. P. (2017). Variation, covariation, and functions: Foundational ways of thinking mathematically. In J. Cai (Ed.), *Compendium for research in mathematics education* (pp. 421-456). Reston, VA: National Council of Teachers of Mathematics.