AN ANALYSIS OF STUDENTS' MATHEMATICAL MODELS FOR MUSIC

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This paper describes a modeling task designed to improve students' understanding of music and related unit structures (e.g., whole note, half note). Fourteen upper elementary students were asked to build models of melodies using Cuisenaire rods and make arguments about how their models represented what they heard. Our analysis of students' models suggested four categories of models. Students exhibited one- or two-dimensional reasoning with either (or both) height and length correspondence that varied in terms of duration and/or pitch features.

Keywords: Modeling, Representations and Visualization, Measurement

Background and Literature Review

Mathematical modeling focuses the relevance of mathematics through the use of authentic contexts where students use their mathematics to solve relevant problems (COMAP & SIAM, 2016). There is a growing emphasis on the inclusion of mathematical modeling in school mathematics (e.g., National Council of Teachers of Mathematics, 2000; National Governor's Association Center [NGAC] & Council of Chief State School Officers [CCSSO], 2010). While the phrase *mathematical modeling* has been used in many ways, we consider the description of mathematical modeling from the Common Core State Standards, which describes modeling as "the process of choosing and using appropriate mathematics and statistics to analyze empirical situations, to understand them better, and to improve decisions" (NGAC & CCSSO, 2010). In this description, the main focus of mathematical modeling is learning to make decisions and assumptions when interpreting a real-world scenario using a mathematical lens. These scenarios are often posed using open-ended tasks where students have the freedom and flexibility to create their own non-prescribed models (COMAP & SIAM, 2016). Because mathematical modeling requires creativity and allows for varied solution strategies, modeling tasks inherently provide multiple entry points and differentiation opportunities (Cirillo et al., 2016).

Prior research studies showed that mathematical modeling tasks were helpful in revealing student thinking and that modeling tasks enable students of differing performance levels to interpret, invent, and find solutions (e.g., Aguilar Battista, 2017; Carmona & Greenstein, 2007; Koellner-Clark & Lesh, 2003; Mousoulides, Pittalis, Christou, & Sriraman, 2010). Despite the existing literature on mathematical modeling, there is a need for further research in the elementary grade levels. An analysis of 29 articles (published between the years 1991-2015) that focused on elementary mathematical modeling (ages 10 and below) revealed that more research (as well as teacher training) related to mathematical modeling in the elementary grades is needed (Stohlman & Albarracin, 2016).

In the modeling task that we share in this report, students are expected to use "the language of mathematics to quantify real-world phenomena and analyze behaviors" (COMAP & SIAM, 2016, p. 8). The real-word phenomena is the representation of musical notes. We chose to develop a modeling task for music because musical notes are inherently mathematical due to the proportional relationship of their size (i.e., duration of each note). Additionally, integrating music and mathematics appears to be a particularly effective intervention for students to improve students' conceptual understanding of

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fractions, especially for high needs students (Courey et al., 2012). In order to contribute to the understanding of framing instruction with modeling tasks in earlier grades, we focused on the following research questions in our study: *What were the mathematical assumptions and decisions students made when creating physical models to represent musical melodies? What were the underlying mathematical characteristics of their models and were there any similarities and/or differences between models?*

When learning a mathematical concept, children use actions. While these actions can initially be physical or mental, ultimately, the actions are mental that may or may not have been derived from physical actions or words (Sarama & Clements, 2009). When creating our own models during the design phase of the task, we determined that our own mental actions included unitizing: defining a unit and a sub-unit (i.e., whole and half notes). Unitizing is defined as "the process of constructing chunks in terms of which to think about a given commodity" (Lamon, 2012, p. 104). Because unitizing is a subjective process, encouraging flexibility and highlighting the relationship between unitizing and understanding fractions and equivalence is important (Lamon, 2012). We focused on students' unitizing mental actions while analyzing their models.

Methodology

The motivation for the *Modeling Music* task was to utilize the multiple ways in which music can be represented to emphasize the proportional relationship of musical notes. To show the different representations of music as well as how these different representations are related, we developed a framework which had the components of song, sound wave, sheet music, and physical tools. This particular modeling task attended to the bi-directional relationships between melody, sheet music, and physical tools representations.

Four melodies (Melodies A, B, C, and D) were created and then purposefully sequenced to highlight differences in the length of the notes (Figure 2). The first two melodies (A and B) were solely comprised of either whole or half notes. The third melody (C) was a combination of whole and half notes and the fourth melody (D) was a combination of whole, half, and quarter notes.

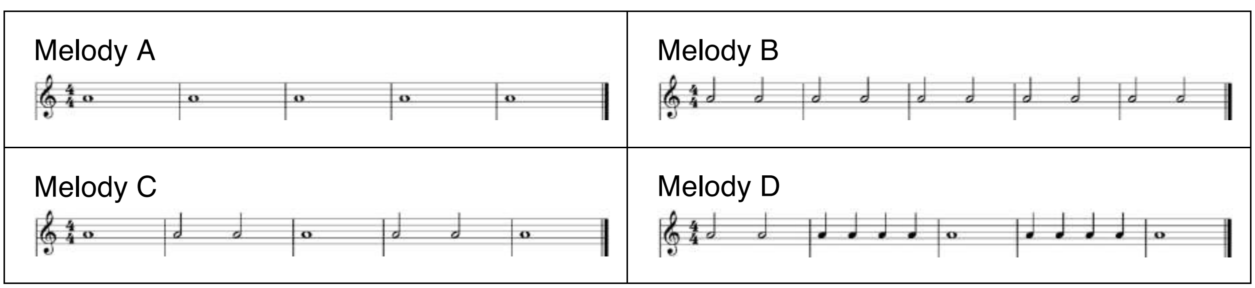

Figure 1. Sheet music for Melodies A, B, C, and D

Participants and Implementation

Fourteen upper-elementary (fourth and fifth grade) students participated in the *Modeling Music* task during a summer ice skating camp in July 2019. The daily schedule of the camp limited the time allotted for the *Modeling Music* task to 45 minutes and as a result, students were only able to create models for the first three melodies. The activity sequence for the *Modeling Music* task consisted of three parts: (1) listening to the melody, (2) recording and sharing notices and wonders about the melody, and (3) building the model using Cuisenaire rods. Students were not provided with any guidance or direction when building their models, which required them to make their own assumptions and decisions during the modeling process, as well as identify the underlying mathematical relationships in their models.

Data Collection and Analysis

In order to better understand students' modeling strategies, the data we collected during task implementation included students' individual written responses to the notices and wonder prompts for each melody, a written record of students' verbal descriptions of each melody, and photographs of the Cuisenaire rod models students created for each melody. The students' models and their written descriptions were analyzed using comparative analysis (Merriam, 1998). The similar models were first categorized into similar chunks (e.g., models using one-dimensional reasoning). In the next revision, this classification was elaborated into more defined categories and we looked for the unitizing structures involved in the models. We used measurement ideas to analyze the multiple representations of proportional relationships and we used basic principles of measurement (e.g., relating size and units) to explore how these relationships were connected within the context of music.

Results and Discussion

Students' notices and wonders for each of the melodies highlighted several common themes. Some of these themes revealed the underlying mathematics students observed (e.g., distance between notes, length of notes). Other themes revealed students' perceptions of the sound (e.g., pitch, tempo). Students' Cuisenaire rod models of the melodies revealed their modeling strategies, including the assumptions and decisions they made for mathematizing the melodies.

Modeling Single Note Melodies (Melody A and B)

When modeling single note melodies, students built either a single rod model or a collection of rods model to represent one note (see Table 1). The main difference between these models was how students decided to represent one unit. With the single rod model, students decided to define one note with one rod, whereas with the collection of rods model, students decided to define one note with a collection of rods in a staircase shape. With both the single rod model and collection of rod models, students assumed that the notes in the melody were identical and chose to iterate their unit to reflect this assumption.

Table 1: Student Models of Single Note Melodies (Melody B)

Single Rod Model	Collection of Rods Model
	

Modeling Two-Note Melodies (Melody C)

When modeling the two-note melody, students had to decide how to represent both whole and half notes in a single model. Students' models were categorized based on which characteristics of the rods they attended to when representing the different notes as summarized in Table 2.

Table 2: Categories for Two-Note Melodies (Melody C)

Category and Sample Model	Defining Characteristics
(1) 1-D: Length Correspondence (Duration) 	Attended to rod length to represent each note. Length of half note (red) corresponded to length of whole note (purple).
(2a) 2-D: Height Correspondence (Duration) 	Attended to horizontal length (number of rods) and height (length of rods) to represent each note. Length of starting rod of half note (yellow) corresponded to length of starting rod of whole note (orange).
(2b) 2-D: Length Correspondence (Duration) 	Attended to horizontal length (number of rods) and height (length of rods) to represent each note. Number of rods representing each note had a 4:2 proportion.
(3) 2-D: Height, Length Correspondence (Duration) 	Attended to horizontal length (number of rods) and height (length of rods) to represent each note. Length half note (yellow) corresponded to length of starting rod of whole note (orange) and number of rods representing each note had a 2:1 proportion.
(4) 2-D: Height, Length Correspondence (Duration and Pitch) 	Attended to horizontal length (number of rods), height (length of rods), and pitch (starting rod) to represent each note. Number of rods representing each note had a 4:2 proportion. Used same starting rod for both whole and half notes.

Conclusion

The *Modeling Music* task clearly provided students with multiple entry and exit points as evidenced by the sheer variety in students' models. In addition, unpacking students' mental actions when building their models revealed commonalities in students' thinking related to unitizing and proportional reasoning (e.g., half/whole note relationships). Our analysis provided a method of categorizing students' models based on their defining characteristics, which brought to light the assumptions and decisions made by students during the modeling process.

Research related to students' mathematical modeling strategies provides opportunities for rich descriptions of student thinking. Our findings are promising in terms of further study of modeling tasks and the value of using modeling tasks to explore students' reasoning and strategies, including application of prior knowledge, when solving open-ended problems. The *Modeling Music* task also suggests a framework for task design and model categorization that can allow for further mathematical modeling research in the elementary grades.

Our findings can also inform instructional decisions. Having a framework for model categorization (in terms of underlying mental actions) allows us to anticipate student thinking, which can help educators better prepare instruction related to both mathematical modeling and the development of measurement concepts.

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