ERROR PATTERNS IN PROSPECTIVE K-8 TEACHERS' POSING OF MULTI-STEP ADDITION AND SUBTRACTION WORD PROBLEMS

Ali Foran
Texas A&M University
aliforan@tamu.edu

Rachael M. Welder
Texas A&M University
welder@tamu.edu

National and state standards in the US have emphasized the importance of solving and posing word problems in students' mathematics learning for decades. Therefore, it is essential for prospective teachers (PTs) to have the mathematical knowledge necessary to teach these skills to their future students. Unfortunately, little research has investigated how PTs develop problem-posing skills. This study investigated PTs' abilities to pose two-step addition and subtraction word problems in the context of a collegiate teacher education course. The researchers analyzed incorrect problems to identify error patterns among the mistakes made by PTs. By employing thematic qualitative text analysis, the researchers identified eight distinct common error categories. These results can be used to inform teacher education and to adapt tasks and instructional strategies for more effectively helping PTs develop their problem posing abilities.

Keywords: Teacher Education - Preservice, Mathematical Knowledge for Teaching, Instructional Activities and Practices, Number Concepts and Operations

Introduction

The Standards for Preparing Teachers of Mathematics put forth by the Association of Mathematics Teacher Educators (AMTE, 2017) call for beginning teachers of mathematics to "regard doing mathematics as a sense-making activity that promotes perseverance, problem posing, and problem solving. In short, they exemplify the mathematical thinking that will be expected of their students," (p. 9). The standards further indicate that effective mathematics education programs "develop positive dispositions toward mathematics, including persistence and a desire to engage in posing and solving problems," (p. 70). As such, helping prospective teachers (PTs) learn how to pose mathematics word problems should be a goal of teacher preparation programs. Towards this goal, the researchers investigated two research questions:

1. With what frequency are PTs able to write correct, two-step addition and subtraction word problems?
2. What patterns emerge in the errors that arise when PTs write two-step addition and subtraction word problems?

The researchers of this study evaluated addition and subtraction word problems posed by K-8 PTs enrolled in an undergraduate mathematics problem-solving course with the intent of identifying emergent trends of conceptual difficulties. The results of this study can inform mathematics teacher educators (MTEs) in developing targeted and meaningful activities to address PTs' difficulties and support their learning to pose multi-step word problems.

Literature Review

Even though problem posing has been discussed in mathematics education since the 1980s (Brown & Walter, 1983; Kilpatrick, 1987; NCTM, 1989), little has been done to investigate or ensure that teachers are prepared to pose problems to their students. Researchers have dug deeply into the role problem-posing can play in students' mathematical development (Akay & Boz, 2009; Alibali et al., 2009; Sharp & Welder, 2014; Bonotto, 2013; Silver & Cai, 1996; Ticha & Hospesova, 2013), and this work has found that problem-solving skills do not necessarily equate to problem-posing skills.

In: Sacristán, A.I., Cortés-Zavala, J.C. & Ruiz-Arias, P.M. (Eds.). (2020). *Mathematics Education Across Cultures: Proceedings of the 42nd Meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education*, Mexico. Cinvestav / AMIUTEM / PME-NA. https:/doi.org/10.51272/pmena.42.2020

Furthermore, Alibali and colleagues found that having students write original multiplication word problems "revealed difficulty with the underlying meaning of multiplication" (p. 257). Similarly, Sharp and Welder (2014) found that asking seventh graders to write a division of fractions story problem exposed multiple "areas of limited conceptions that may not have been identified through traditional algorithm-driven assessments" (p. 546).

Since problem posing is an instructional activity that has been shown to benefit student learning by providing instructors insight into students' conceptions, teachers should be prepared to incorporate such activities in their future classrooms. However, to do so, they must first learn how to pose problems themselves. Therefore, it is important that PTs be given opportunities to develop problem-posing skills during their mathematics education preparation.

As MTEs, our attention then turns to how we can efficiently prepare PTs to pose a variety of word problems. The National Research Council tells us that "addition and subtraction are used to relate amounts before and after combining or taking away, to relate amounts in parts and totals, or to say precisely how two amounts compare" (2009, p. 32). Their work and the Common Core State Standards Initiative (NGA & CCSSO, 2010) have highlighted the multiple situations in which addition and subtraction occur by developing a framework for word problems that can be used to categorize them according to their structural differences. This framework produced 14 clearly distinguished categories, which give way to instructional strategies that MTEs can use for guiding PTs in developing problem-posing skills.

Methods

At a tier one research institution in the southern United States, the researchers collected data from PTs enrolled in an undergraduate mathematics problem-solving course that focuses on teaching mathematics through problem solving (Alwarsh, 2018; Bostic et al., 2016; Chapman, 2017; Fi & Denger, 2012). Instructors of this problem-solving course have incorporated a variety of instructional activities and strategies to support PTs in their learning to create original one- and multi-step addition and subtraction word problems that utilize a variety of problem structures. One such task asked PTs to pose four 2-step addition and subtraction problems to match four sets of specified structures (e.g., *change – add to – change unknown and part-part-whole – part unknown*). The data analyzed in this report includes problems posed by PTs in one instructor's course across two semesters. Thirty-seven PTs were enrolled in each semester of the class for a total of 74 PTs. All PTs were enrolled in programs leading to teacher certification in the areas of EC-6 (Generalist) or grades 4-8 mathematics and science, English, or history.

Throughout their coursework, PTs were introduced to the taxonomy of common addition and subtraction situations as identified by the Common Core State Standards (NGA & CCSSO, 2010) as a basis for discussing structural differences between addition and subtraction word problems. After categorizing and solving a variety of one-step word problems, the PTs posed one-step word problems to match each of the 14 possible problem structures. PTs received feedback on the one-step problems they wrote, which were mostly correct. Afterwards, class activities focused on categorizing and solving two-step addition and subtraction word problems. Lastly, PTs were given the aforementioned assignment in which they were instructed to pose four two-step word problems to match given pairs of addition and subtraction problem structures. This assignment resulted in 282 PT-posed, two-step word problems (n=282).

To analyze the 282 word problems, the researchers used thematic qualitative text analysis. First the problems were coded as being correct (n=124) or incorrect (n=158). Next, categories of error patterns identified in the 158 incorrect problems were created both deductively and inductively (Kuckartz, 2014), first at a macro and then micro levels. A temporary category was created based on the number of steps required to solve each incorrect problem, followed by whether the structures of the posed problem matched the assigned structures. As the researchers continued their analyses, they determined whether each new incorrect problem matched an existing category or if a new category was necessary. To test the validity of the devised coding scheme, the researchers re-coded all 282 word problems independently according to their correctness and the categories of error patterns that had emerged. After discussion, agreement was reached for 100% of the analyzed word problems. Lastly, the categories of error patterns were analyzed by the frequency at which they were exhibited by the PTs.

Results

As mentioned, 124 of the 282 word problems correctly provided a scenario that included two situations that matched both of the assigned structures and asked a question that required a two-step calculation utilizing the unknowns from each situation. Fifty-six of the remaining 158 problems correctly posed a valid two-step addition/subtraction question and were only deemed incorrect in this analysis because they simply did not meet the structural criteria of the prompts.

The remaining 102 PT-submitted problems were deemed as having one or more structural errors. The analysis of these errors led to the identification of eight distinct categories of error patterns, dependent upon the number of steps required to solve the problem, the appropriateness of the question(s) asked, and the use of the assigned structures. Due to space restrictions in this report, we will explicate only the most-frequent category of error pattern, problems that only required one step (n=68), and provide examples of the ways in which one-step errors occurred as found in PTs' work. Of the remaining 34 problems, 12 required more than two steps, three required zero steps (as the solutions had been provided within the context of the problems), four required algebra in their solutions, and 15 could not be solved with the given information.

One-step Problems (n=68)

The most common error resulted from PTs who were able to build up two addition/ subtraction structural situations but did not properly utilize the unknown information from the first scenario to form a question that required a two-step calculation (n=68). Sixty-three of these *one-step* problems posed a single question, but the question still failed to connect the two unknowns. These 63 *one-step – one question* problems were further categorized according to whether the PTs used the assigned structures or not. The remaining five *one-step* problems resulted in the posing of two separate questions. Examples of each type are provided below.

One-step – one question – two correct structures (n=31). Thirty-one of the 63 *one-step, one-question* problems built contextual scenarios that correctly matched both of the requested problem structures. However, due to a lack of connection between the two scenarios, the question posed only required one calculation to be solved. For example, one PT submitted the following problem, exhibiting this common error, in response to the second prompt (*change – add to – start unknown; compare – more – bigger unknown*):

> Sarah had some pieces of candy. Four more pieces were given to her, so she had ten pieces of candy total. Amanda had five more pieces of candy than the amount of candy Sarah was given. How many pieces of candy does Amanda have?

This PT provided a scenario that matched the two assigned structures but did not pose a question that would require the solver to utilize the unknown information from the first step as known information in the second step. Specifically, in this example, the unknown information in the first scenario is the number of pieces of candy Sarah starts with, but the second scenario connects the second unknown to the number of pieces Sarah was given. Since this information was provided ("*Four more pieces were given to her*"), the only required step to answer the question is adding four and five to get nine pieces of candy.

One-step – one question – incorrect structure(s) (n=32). The remaining 32 of the 63 *one-step, one-question problems* exhibited the same error as above but were further flawed in the sense that the scenarios posed did not fully match the requested structures. For example, for prompt 3 (*part-part-whole – addend unknown; compare – more – difference unknown*), one PT submitted the problem:

> Sarah has four red shirts and some green shirts. Sarah has two more red shirts than green shirts. How many green shirts does she have?

Again, in this problem, only one calculation is necessary to answer the question posed: four (red shirts) minus two (more red shirts than green shirts) equals two (green shirts), making it a *one-step* problem. However, this problem also exhibits structural issues. The first scenario (*part-part-whole – part unknown)* was not fully developed, as the whole amount was never provided (leaving two pieces of unknown information: the number of green shirts and the total number of shirts). Furthermore, the second scenario created a *compare – more – smaller unknown* situation (when the prompt specified *difference unknown*).

One-step – two questions (n=5). A small subgroup of these *one-step problems* (n=5) contained two independent scenarios and asked two separate questions in an effort to satisfy the two-step prompts. Four of these problems included the correct assigned structures, one did not. The PTs who wrote these problems knew that two steps were necessary to satisfy the given task but showed difficulty in connecting their unknowns into a single question. For example, one PT-submitted the following problem to the third prompt (*part-part-whole – addend unknown; compare – more – difference unknown*):

> Maria has 4 apples and some cherry pies. She has a total of 7 pies. Kara has more pies than Maria. They together have a total of 15 pies. How many cherry pies does Maria have, and how many pies does Kara have?

Discussion

The skill analyzed in this study was the ability of PTs to write addition and subtraction word problems that utilized one unknown in a second scenario to create a two-step problem. The level of difficulty of these problems for solving purposes is quite low, but the necessary skill to create such problems proved quite high with only 44% of the PTs able to correctly formulate a two-step word problem as requested. Alarmingly, a large proportion of the 56% of PTs who were unable to create two-step problems tended toward writing one-step problems with disconnected or incorrectly connected scenarios.

Teachers at every level need to be prepared to create original word problems and support their students in developing problem-posing skills. Our findings are especially concerning given that the participants in this study were being trained to teach mathematics in elementary classrooms yet displayed great difficulty in formulating elementary-level word problems. As we, the researchers, apply the findings to our mathematics problem-solving course, we are using the error pattern framework developed here to facilitate targeted discussions of common errors in the classroom. The findings of this study will inform our development of a task designed to confront common errors head on so that PTs will be more cognizant of effective and ineffective problem-posing strategies. Future research will study the effects of this task.

References

Akay, H. & Boz, N. (2009, January). *Prospective teachers' views about problem-posing activities*. Paper presented at the annual meeting of the World Conference on Educational Sciences, Nicosia, North Cyprus.

Alibali, M. W., Brown, A. N., Stephens, A. C., Kao, Y. S., & Nathan, M. J. (2009). Middle school students' conceptual understanding of equations: Evidence from writing story problems. WCER Working Paper No. 2009-3. *Wisconsin Center for Education Research (NJ1).* Retrieved from http://eric.ed.gov/?id=ED506072

Alwarsh, A. A. (2018). Productive mathematical discussions in teaching through problem solving. *Ohio Journal of School Mathematics*, *78*, 4-10.

Association of Mathematics Teacher Educators (AMTE). (2017). *Standards for preparing teachers of mathematics.* Available online at amte.net/standards

Bonotto, C. (2013). Artifacts as sources for problem-posing activities. *Educational Studies in Mathematics*, *83*, 37-55.

Bostic, J. D., Pape, S. J., & Jacobbe, T. (2016). Encouraging sixth-grade students' problem-solving performance by teaching through problem solving. *Investigations in Mathematics Learning*, *8*(3), 30-58.

Brown, S. I. & Walter, M. I. (1983) *The art of problem posing*. Philadelphia, PA: Franklin Institute Press.

Chapman, O. (2017). High school teachers' pedagogical conceptions that support teaching through problem solving. In E. Galindo & J. Newton (Eds.), *Proceedings of the 39th annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education* (pp. 609-612). Indianapolis, IN: Hoosier Association of Mathematics Teacher Educators.

Fi, C. D. & Degner, K. M. (2012). Teaching through problem solving. *The Mathematics Teacher*, *105*, 455-459.

Kilpatrick, J. (1987). Formulating the problem: Where do good problems come from? In A. H. Schoenfeld (Ed.), *Cognitive Science and Mathematics Education* (pp. 123-147). Hillsdale, NJ: Lawrence Erlbaum Associates.

Kuckartz, U. (2014). *Qualitative text analysis: A guide to methods, practice and using software*. London: Sage Publications.

National Council of Teachers of Mathematics (NCTM). (1989). *Curriculum and evaluation standards for school mathematics*. Reston, VA: Authors.

National Governors Association Center for Best Practices and the Council of Chief State School Officers (NGA & CCSSO). (2010). *Common core state standards for mathematics: Common addition and subtraction*. Retrieved from http://www.corestandards.org/Math/Content/mathematics-glossary/Table-1/

National Research Council (NRC) (2009). *Mathematics learning in early childhood: Paths toward excellence and equity*. Washington, DC: The National Academies Press.

Sharp, J. & Welder, R. M. (2014). Reveal limitations through fraction division problem posing. *Mathematics Teaching in the Middle School*, 19(9), 540-547. DOI: 10.5951/mathteacmiddscho.19.9.0540

Silver, E. A. & Cai, J. (1996). An analysis of arithmetic problem posing by middle school students. *Journal for Research in Mathematics Education*, *27*, 521–539.

Texas Education Agency (TEA). (2012). *19 TAC Chapter 111. Texas essential knowledge and skills for mathematics*. Retrieved from http://ritter.tea.state.tx.us/rules/tac/chapter111/index.html

Ticha, M. & Hospesova, A. (2013). Developing teachers' subject didactic competence through problem posing. *Educational Studies in Mathematics*, *83*, 133-143.