GESTURE IN PROOF AND LOGICAL REASONING

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Twelve doctoral students in mathematics took part in clinical interviews during which they were asked about their experiences with teaching, learning and doing proof. They were also asked to work together to find a proof for an unfamiliar conjecture. The students' discourse, including gesture, was analysed from the perspective of embodied cognition. In particular, a potential continuity between mathematical and everyday discourse was investigated, with a particular focus on epistemic conditionals, that is, "if-then" statements.

Keywords: Embodiment and Gesture; Reasoning and Proof; Advanced Mathematical Thinking; Cognition

Objectives of Study

In recent decades, researchers have investigated how the body in implicated in mathematical teaching and learning, challenging the paradigm that cognition is amodal and abstract, based solely "in the head." In addition, attention to embodiment has broadened the focus within mathematics education research beyond written symbols, images, and oral speech to include modalities such as gesture and other bodily movements (Edwards, Ferrara, & Moore-Russo, 2014; Hall & Nemirovsky, 2012). The purpose of this paper is to examine mathematical proof and logical reasoning from the perspective of embodied cognition (Edwards, 2011; Varela, Thompson, & Rosch, 1991), using data collected from clinical interviews with 12 doctoral students in mathematics.

The analysis presented here is based on the principle of cognitive continuity; that is, the proposition that there are not multiple different kinds of thinking, even within a domain like mathematics, but rather all thought is ultimately founded in embodied, physical experience (Johnson, 2012; Lakoff, & Núñez, 2000; Varela, Thompson, & Rosch, 1991). The implication is that even with "advanced" mathematical thinking, like that involving proof and logic, connections can be made with more everyday kinds of thinking and basic human experiences. As Johnson states, "we do not have two kinds of logic, one for spatial-bodily concepts and a wholly different one for abstract concepts. There is no disembodied logic at all. Instead, we recruit body-based, image-schematic logic to perform abstract reasoning" (Johnson, 2012, p. 181). The research reported her aims to delineate one way in which elements within the abstract domain of mathematical proof are connected to similar ones in everyday discourse.

Following Hanna (1990), we take proof to be:

> [A] finite sequence of sentences such that the first sentence is an axiom, each of the following sentences is either an axiom or has been derived from preceding sentences by applying rules of inference, and the last sentence is the one to be proved. (Hanna, 1990, p. 6)

Although this definition is appropriate for the end product of a process of proving, we also frame proof and proving as a specialized type of discourse, built on simple logical elements and constrained by agreements on validity generated within the mathematical community. In the current research, the specific focus is on logical statements that take the form of "if-then" statements; these statements can be seen as the building blocks of proofs. The central research question is whether the physical gestures that accompany these "if-then" statements when talking about proof are similar to those accompanying "if-then" statements in non-mathematical contexts. If so, then this would provide support for the notion of cognitive continuity between these two contexts.

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Theoretical Perspective

The research was carried out utilizing the theoretical perspective of embodied cognition, making use of tools from cognitive linguistics and gesture studies. The theory of embodied cognition focuses on the bodily basis of thinking, that is, "on the ways in which complex adaptive behavior emerges from physical experience in biologically-constrained systems" (Núñez, Edwards, & Matos, 1999, p. 49; see also Varela, Thompson, & Rosch, 1991). Here, we focus not on specific mathematical content, for example, algebra or analysis, but on the mechanisms used by mathematicians to test and establish logical truth. Under Johnson's continuity principle, we propose that deductive proof and logic are constructed using the same basic conceptual building blocks as more mundane thought (Johnson, 2007).

Within an embodied cognition framework, mathematics is not seen as a transcendental, formal collection of rules and patterns, unrelated to everyday thinking and experience, but instead, as a human intellectual product, one which develops both historically as a discipline over time, and ontologically as it is constructed by an individual learner. It is socially-constructed, but not in an arbitrary way, being both constrained and enabled by the biological capabilities and physical situatedness of human beings. Embodiment does not deny the influence of social interaction and culture; rather it grounds it in shared biological constants (Hall & Nemirovsky, 2012; Nuñéz, Edwards, & Matos, 1999). As stated by Hall and Nemirovsky (2012), "We think of concepts (in mathematics but also in other domains) as forms of modal engagement in which bodies incorporate and express culture" (p. 212).

Prior Research on Proof

Prior research has been fruitful in its examination of the learning and teaching of proof, whether addressing the understandings and misunderstandings of novices, productive instructional practices and tools, or the thinking of advanced mathematicians (a selection of recent work can be found in Lin, Hsieh, Hanna & deVilliers, 2009). The current research builds on this foundation, particularly in seeing proof as a form of socially constructed knowledge and a specific form of discourse (Balacheff, 1991; Sfard, 2001). The current analysis adds the lens of embodiment and gesture studies in analyzing this discourse.

Prior Research on Conditional Statements

From the point of view of cognitive linguistics, mathematical or logical deductions ("if-then" statements) belong to a linguistic category known as conditionals (Dancygeir & Sweetser, 2005). Specifically, "if-then" statements represent the type called epistemic conditionals, because they reference a reasoning process, rather than a prediction or statement of fact. Two examples of epistemic conditionals are: "If the car is in the driveway, he must be home" and "If x is even, then x/2 is an integer" (p. 17). These kinds of conditionals involve what Danceygeir and Sweetser call a "metaphoric 'compulsion'" (p. 20) in which the speaker is "forced" to draw the given conclusion, either based on inductive reasoning ("the car is almost always in the driveway when he is home") or deductive logic (the mathematical definition of "even"). An analysis of how this metaphoric "compulsion" is grounded in early embodied experiences, providing a physical basis for the later construction of the notion of proof, can be found in Edwards (2017, 2019).

In addition to the linguistic analysis of explicit conditional statements by Danceygeir and Sweetser, recent research by Sweetser has examined gestures associated with spoken conditionals. In a study involving 402 video clips of talk shows, Sweetser and Smith (2015) found that conditionals were generally accompanied by a particular hand motion, specifically a movement along a transverse axis through gesture space, starting on the speaker's left and moving toward the speaker's right. The current analysis examined the gestures of mathematical doctoral students to see whether they also reflected this characteristic motion when orally stating epistemic conditionals. If so, then this would constitute evidence of the continuity between everyday uses of conditionals and their use in mathematical proof.

Methods

The research took the form of a qualitative study similar in format to a task-based clinical interview, recorded on audio and videotape. The participants, pairs of doctoral students in mathematics, were first interviewed about their specializations in mathematics, their experiences with teaching proof, and their ideas about whether there are different kinds of proofs. They were then presented with the conjecture below on a sheet of paper, and asked to work together to find a proof for it.

> Let *f* be a strictly increasing function from [0, 1] to [0, 1]. Prove that there exists a number *a* in the interval [0, 1] such that f(*a*)=*a*.

They were given 40 minutes to try to find a proof, during which the researcher left the room so that the participants could work without feeling self-conscious about being observed. During the third part of the interview, the students were asked to evaluate a visual "proof." The results presented here were drawn from the first part of the interview.

Participants

The participants were 12 doctoral students in mathematics, 9 men and 3 women, attending a research university in the United States. They were placed in pairs for the interviews based on their availability and schedules. They all knew each other as fellow students in the doctoral program, and two of the women, who worked together as a pair, were good friends. The time they had spent in the doctoral program ranged from less than a year to almost four years, and all had had experience in teaching undergraduate mathematics courses, although this experience did not involve much teaching of proof.

Context

The interviews took place in a small unused office with a blackboard at one end. The participants sat on chairs in front of the blackboard, facing the interviewer and the video camera. They were asked to use only the blackboard while working on the proof.

Data Collection

All sessions, lasting from 60 to 90 minutes each, were recorded on videotape and via digital audio, with the camera oriented to capture both the blackboard and the students as they sat or stood in front of it. A total of 6 hours and 55 minutes of video and audio were collected.

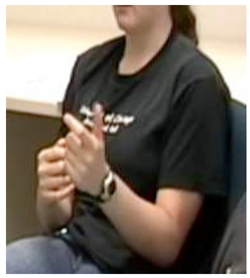
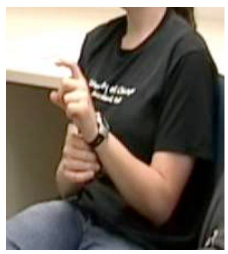
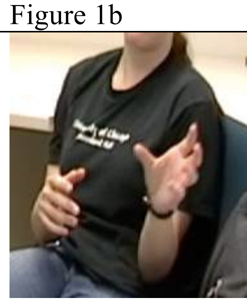
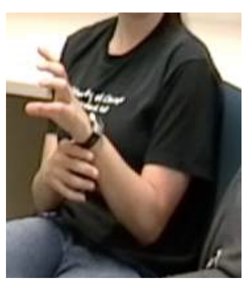
Analysis

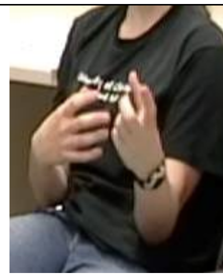
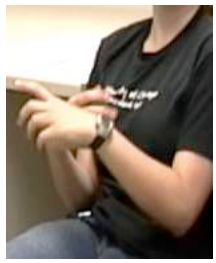
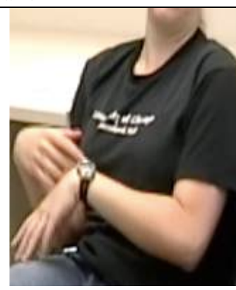
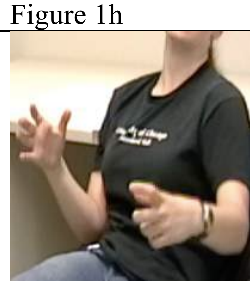
The audiotapes were transcribed and annotated with brief notations of gestures as well as time spans of the use of different modalities by the participants. Specific segments of discourse containing gestures of interest were analyzed in more detail, utilizing the concurrent speech, written symbols, and drawn graphs to develop plausible interpretations consistent the context and with other research into gesture (Alibali, Boncoddo, & Hostetter, 2014; McNeill, 1992; Perrill & Sweetser, 2004).

Results

The analysis of the doctoral students' gestures when making conditional statements did indeed reveal the presence of the same left-to-right transverse gesture previously identified in non-mathematical contexts. Although the use of epistemic conditionals in speech was found throughout the video data, most instances occurred while the students were actively working on finding a proof; thus, their hands were often occupied with chalk or they were pointing to inscriptions on the board, meaning that "if-then" statements were often not accompanied by gestures. However, the transverse gesture did occur regularly in the data, approximately once in every ten instances in which an epistemic conditional was uttered. This occurred primarily when the students were talking to the interviewer, explaining a proof.

The example shown in Figure 1 illustrates three instances of this gesture form. In this example, the epistemic conditional that the student is expressing can be summarized as follows: "If you have a scalar function and a vector function, then the rule for finding their product is the same as the rule for finding the product of two scalar functions."

AC: Well, I guess, so, the other day they were trying to prove that, um, if you have some scalar function of T Int: Uh huh AC: —and some vector function of T,	 Figure 1a  Figure 1b	Left hand starts in horizontal C-shape ("bracket") facing upward on left side of body Left to right motion with left hand along transverse axis, ending in middle of body, with C-shape turning vertical
Int: Uh huh AC: —that the derivative of their product...	 Figure 1c  Figure 1d	Left to right motion with left hand along transverse axis, with left hand open and facing outwards. Left hand begins on left side of body and ends in middle of body.
is the same...		Rapid left to right motion with left hand along transverse axis. Left hand starts in loose horizontal C-shape ("bracket") facing upward on left side of body and ends in pointing gesture to the right.

	 Figure 1e  Figure 1f	
AC: ...product rule essentially that you know from just, you Int: (talking over): Uh huh.	 Figure 1g  Figure 1h	A complex motion in which the left hand begins by pointing downward, then is moved in a circle twice around the right hand while saying "you know," ending up open and facing the speaker
AC: know from like scalar functions	 Figure 1i	Left hand moves to right and finishes in horizontal C-shape ("bracket") on left side of body. This is the same shape and location as when the phrase "scalar function" was initially uttered.

Note: Underlined speech indicated the stroke or emphasized portion of the gesture

Figure 1: Student's discourse about scalar functions

The sequence of gestures accompanying the student's speech is very rich, taking into account characteristics including hand shape and orientation, hand location, and movement of the hands through space. Consistent with other conditionals used in non-mathematical contexts, the sequence includes left-to-right motion along the transverse axis; in fact, this transverse motion occurs three different times, as shown in the pairs of figures above:

- Figure 1a – b: A relatively small left-to-right motion of the left hand, as AC begins by saying, "If you have some scalar function of T and some vector function of T." This sequence also includes a change in orientation of the left hand; when holding it on the left, AC uses an upward-opening (horizontal) C-shape as if "bracketing" or "holding" a scalar function. As she moves her hand to the right, she rotates her wrist so that when she says, "vector function," the C-shape is now vertical. She thus uses both hand shape and hand location to gesturally distinguish the two different kinds of functions.
- Figure 1c – d: A wider left-to-right motion of the left hand, as AC says, "the derivative of their product." In this case, the hand shape stays the same throughout, open and facing outward.
- Figure 1e – f: After saying "derivative of their product," AC pauses briefly, then makes a very rapid left-to-right motion of her left hand while saying, "is the same," starting with a horizontal C-shape and ending with a right-facing point.

As can be seen above, in addition to an overall left-to-right movement that occurs three times during the sequence, gestures are also used to mark or indicate specific mathematical objects, in a scheme that Calbris (2008) calls "two-entity opposition." Two-entity opposition occurs when either two locations in space or the two hands are used to denote or "mark" two related but distinct entities. In Figure 1, this happens when AC uses a horizontal "bracket" held to her left when saying "scalar functions" and then a vertical bracket held to her right when saying "vector functions." The terms "derivative" and "product" have the same hand shape but are marked by left and right hand locations, indicating two-entity opposition.

The discourse segment ends with AC discussing a "product rule" while using an iterative circular gesture during a pause in speech. This pause and rhythmic circular gesture may indicate that the participant is searching for her next words (Lucero, Zaharchuk, & Casasanto, 2014). She compares this product rule to a presumably familiar rule for scalar functions. Interestingly, the final gesture of the sequence, associated with the words "scalar function" has an identical shape and location as the gesture used the first time the words were uttered. This is an example of using specific hand shapes and locations in gesture space to "hold" a referent in discourse (Calbris, 2008; McNeill, 1992).

Discussion

Calbris (2008) has stated that in gesture space, the transverse axis can represent logico-temporal concepts, such as cause and effect, or before and after:

> A path in space or time is depicted by a left-to-right movement. But give that body symmetry allows this axis to account for splitting in two as well as two-entity oppositions, it can be used to oppose past and future, or precedence and successor, by locating the past on the left side and the future on the right side. (Calbris, 2008, p. 43)

In the current case, and in the research by Sweetser and Smith (2015), the transverse axis is used to indicate the premise followed by the conclusion of a conditional "if-then" statement.

The transverse axis of the body has been also called "the axis of reading and writing, pointing to the right in the Western world" (Calbris, 2008, p. 28). In this case, the motion of AC's gestures is consistent both with the placement of the "cause" (premise) on the left and the "effect" (conclusion) on the right, as well as the left-to-right order in which premise and conclusion are generally written in English. In the example given above, the left-to-right motion along the transverse axis is thus consistent both with how "if-then" statements are written in English, and with prior research and theory identifying this gestural motion with logical and conditional statements.

Taken in conjunction with related research (Edwards, 2010, 2011, 2017), we would argue that the examples above provide further evidence that proof and its building blocks, statements of logical deduction, are not abstract elements of disembodied rationality. Instead, we argue, these sophisticated forms of discourse make use of metaphorical mappings related to motion, and are supported by conceptual metaphors grounded in physical experiences.

Mathematical proof is thus seen as a specialized cultural product and a specific form of discourse, with particular constraints that distinguish it from everyday speech and make it more powerful for the purpose of exploring structure and patterns. Yet the form that this discourse takes is not arbitrary, but rather is grounded in embodied human experience. As shown above, there exists a continuity between the gestural grounding for the logical conditionals used in proof and those used in non-mathematical contexts. This kind of analysis is relevant to mathematics education because the conceptual sources that students draw from in constructing new mathematical knowledge may not correspond to the more sophisticated intra-mathematical sources that their instructors use (c.f., Núñez, Edwards & Matos, 1999). For example, students who are beginning to learn about formal logic often "import" expectations about conditionals from everyday speech, assuming that "if A, then B" implies "if not A, then not B" (Evans, Newstead, & Byrne, 1993). A better understanding of the cognitive roots of mathematical thinking may help in designing corrective instruction in such situations.

References

Alibali, M. W., Boncoddo, R., & Hostetter, A. B. (2014). Gesture in reasoning: An embodied perspective. In L. Shapiro (Ed.). *The Routledge handbook of embodied cognition.* (pp. 168-177). New Jersey: Routledge.

Balacheff, N. (1991) Benefits and limits of social interaction: The case of teaching mathematical proof. In Bishop A., Mellin-Olsen S., & Van Dormolen J. (Eds.) *Mathematical knowledge: Its growth through teaching* (pp. 175-192). Dordrecht, The Netherlands: Kluwer.

Calbris, G. (2008). From left to right...Coverbal gestures and their symbolic use of space. In A. Cienki & C. Müller (Eds.) *Metaphor and gesture.* (pp. 27-54). Amsterdam, The Netherlands: John Benjamins Publishing.

Dancygeir, B. & Sweetser, E. (2005). *Mental spaces in grammar: Conditional constructions.* Cambridge, UK: Cambridge University Press.

Edwards, L. D. (2010). Doctoral students, embodied discourse and proof. In M. M. F. Pinto & T. F. Kawasaki (Eds). *Proceedings of the 34th Conference of the International Group for the Psychology of Mathematics Education, Vol. 2* (pp. 329 -336), Belo Horizonte, Brazil: PME.

Edwards, L. D. (2011). Embodied cognitive science and mathematics. In B. Ubuz (Ed). *Proceedings of the 35th Conference of the International Group for the Psychology of Mathematics Education, Vol. 2* (pp. 297–304). Ankara, Turkey: PME.

Edwards, L. D. (2017). Proof from an embodied point of view. In B. Kaur, W. K. Ho, T. L. Toh, & B. H. Choy (Eds). *Proceedings of the 41st Conference of the International Group for the Psychology of Mathematics Education, Vol. 2* (pp. 297-304), Singapore: PME.

Edwards, L. D. (2019). The body in/of proof: An embodied analysis of mathematical reasoning. In M. Danesi (Ed.). *Interdisciplinary perspectives on mathematical cognition.* (pp. 119-140). Germany: Springer International Publishing.

Edwards, L. D., Ferrara, F., & Moore-Russo, D. (Eds.) (2014). *Emerging perspectives on gesture and embodiment in mathematics*. Charlotte, NC: Information Age Publishing.

Evans, J. S. B., Newstead, S. E., & Byrne, R. M. (1993). *Human reasoning: The psychology of deduction*. Hove, UK: Lawrence Erlbaum.

Hall, R., & Nemirovsky, R. (2012). Introduction to the special issue: Modalities of body engagement in mathematical activity and learning. *Journal of the Learning Sciences, 21(*2), 207-215.

Hanna, G. (1990). Some pedagogical aspects of proof. *Interchange, 21*(1), 6-13.

Johnson, M. (2008). *The meaning of the body: Aesthetics of human understanding*. Chicago, IL: University of Chicago Press.

Lin, F., Hsieh, F., Hanna, G. & de Villiers, M. (Eds.) (2009). *Proceedings of ICMI STUDY 19: Proof and proving in mathematics education.* Taipei, Taiwan: The Department of Mathematics, National Taiwan Normal University.
Lakoff, G., & Núñez, R. (2000). *Where mathematics comes from: How the embodied mind brings mathematics into being.* New York: Basic Books.
Lucero, C., Zaharchuk, H., & Casasanto, D. (2014). Beat gestures facilitate speech production. In *Proceedings of the Annual Meeting of the Cognitive Science Society* (Vol. 36, No. 36) (pp. 898-903) University of California, Merced: California Digital Library.
McNeill, D. (1992). *Hand and mind: What gestures reveal about thought.* Chicago: Chicago University Press
Nuñéz, R., Edwards, L., & Matos, J. (1999). Embodied cognition as grounding for situatedness and context in mathematics education*. Educational Studies in Mathematics,39*(1-3), 45-65.
Parrill, F., & Sweetser, E. (2004). What we mean by meaning: Conceptual integration in gesture analysis and transcription. *Gesture, 4*(2), 197-219.
Sfard, A. (2001). There is more to discourse than meets the ears: Looking at thinking as communicating to learn more about mathematical learning. *Educational Studies in Mathematics, 46*(1-3). 13-57.
Sweetser, E., & Smith, I. (2015, July). Conditionals, mental spaces and gesture. *Paper presented at the 13th International Cognitive Linguistic Conference*, Newcastle-on-Tyne, England.
Varela, F. J., Thompson, E., & Rosch, E. (1991). *The embodied mind: Cognitive science and human experience.* Cambridge, MA: MIT Press.