FUNCTION IDENTITY AND THE FUNDAMENTAL THEOREM OF CALCULUS

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By analyzing the responses of 100 introductory calculus students to two questions, this study addresses how students understand the fundamental theorem of calculus as it relates to function identity. One question involves students' understandings of the fundamental theorem of calculus, and the other involves their concept definitions of function sameness. This analysis aims to better understand students' concept images of function sameness, both in the context of the fundamental theorem of calculus and in general.

Keywords: University Mathematics, Calculus

The fundamental theorem of calculus (hereafter "the FTC" or "the fundamental theorem") is an important aspect of mathematics that we would like calculus students to understand. The FTC provides a relatively fast way of calculating integrals, which are used in various quantitative situations. One way to view an integral is as a function, say $\int_a^x f(t)dt$, where x is a variable. This is, arguably, the manner in which Newton conceived it (Thompson & Silverman, 2008). In this light, the FTC is actually a statement of function identity: the function g defined by $g(x) = \int_a^x f'(t)dt$ is the same function as h defined by $h(x) = f(x) - f(a)$. This paper addresses student understanding of this concept; specifically, I investigate the following: how do students understand the fundamental theorem in relation to their conceptions of function sameness?

There are three broad topics that apply to this investigation: student understanding of *function*, *function identity*, and the FTC. Research suggests that secondary and university students often do not have a mathematically normative understanding of function (Bardini et al., 2014; Leinhardt et al., 1990; Mirin, 2017; Sfard, 1992; Thompson, 1994; Vinner & Dreyfus, 1989). Function and function identity are intertwined; how a student understands function identity is closely tied to how they understand function (Mirin, 2017). A student's concept of what a function is will be closely tied to how they understand when two functions are identical. For example, if a student thinks of a function as a process, then it would make sense for that student to think of functions as identical whenever they represent the same process. Relatedly, if a student thinks of a function as an equation, then they might therefore think of different but equivalent equations as necessarily representing different functions. Relatedly, some university students struggle with the notion of function identity, classifying functions represented differently as different functions (Mirin, 2017; Mirin, 2018; Mulhuish & Fagan, 2017).

There is little literature on how students understand the fundamental theorem. Thompson (1994) finds that students' issues grasping the FTC are grounded in underdeveloped understandings of rate of change and covariation. Orton (1983) reports the types of mistakes students make in doing problems with definite integrals. He focuses on how students understand definite integrals as limits. However, his study does not address integrals in the context of the fundamental theorem or as functions. Thompson and Silverman (2008) make the point that an integral *as a function* is conceptually different from a definite integral as a number. That is, conceptualizing $g(x) = \int_a^x f'(t)dt$ as a function is different than conceptualizing $\int_a^b f'(t)dt$ for a particular number *b,* in the same way that conceptualizing the squaring function is different from conceptualizing a particular number being squared. In this manuscript, I situate the FTC as a statement about function identity, and hence also as a statement about functions.

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I adopt the constructs described in Tall and Vinner (1981): A student's *concept image* is "the total cognitive structure that is associated with the concept, which includes all the mental attributes and associated properties and processes" (p.152). One component of a student's concept image is their *concept definition,* which is their stated definition of a concept. This study involves investigating student concept definitions for function sameness, while acknowledging that there is likely more to a student's concept image than their stated concept definition.

The overarching epistemology guiding this study is radical constructivism, as described in Thompson (2000). This epistemology takes the perspective that students construct their own mathematical realities. A guiding aspect of my research is to not assume that what is a representation of an abstract mathematical object to us is also viewed as an abstract mathematical object by a student (Thompson & Sfard, 1994). Similarly, what is the same to us (e.g. different representations of the same function) might not be the same to students. This mathematical reality of the students is not directly accessible to us as researchers – the best we can do is create models (explanations) that account for students' responses (Clement, 2000).

Task Design, Subjects, and Data Collection

This is part of a larger study, the first portion of which can be found in Mirin (2018). A quiz was administered by the instructor to 102 students during the last week of an introductory calculus course at Anonymous State University (ASU). The course followed the Stewart (2013) text, and students had, within the week prior, learned about the FTC and practiced textbook problems applying it. The tasks discussed here are in Figure 1 (below).

1. Let p be the function defined on all real numbers by

$$p(x) = \int_2^x 3t^2 dt$$

and let q be the function defined on all real numbers by

$$q(x) = x^3 - 8$$

(a) How are p and q related? (Select option i. or ii.).

i. p and q are the same function.

ii. p and q are not the same function.

(b) Provide an explanation for your answer for 1a

2. Suppose g is a function and h is a function. What does it mean for g and h to be the same function? Explain.

Figure 1: The FTC Question (1) and the Function Sameness Question (2)

The first part of the quiz involved questions regarding derivative at a point of a single function represented in two different ways. The results of that portion indicated that students might have a mathematically non-normative concept image of function sameness. Here, I address student responses to the tasks in the second part of the quiz, which are relabeled as "Question 1", the FTC question, and "Question 2", the function sameness question (Figure 1, above). Notice that Question 1 is an instance of the FTC. The normatively correct response to this question is that p and q are the same function. Question 2 asks the student to give their concept definition of *function sameness.* Note that there are at least two different normatively correct responses to this question; the first is that g and h are the same if and only if g and h share a graph (set of ordered pairs) and also share a codomain, and the second only requires that they share a graph (Mirin et et al., in review). However, codomain was not once mentioned by students.

Analysis and Results

Due to the multiple choice nature of the question, coding the results of the Fundamental Theorem question was straightforward. Two students did not answer the question, nor did they answer 1b or 2. For this reason, they are excluded from the remainder of this analysis, leaving us with a convenient sample size of 100. Of the remaining 100, 61 chose option (i) (that *p* and *q* are the same function), and 39 chose option (ii) (that *p* and *q* are not the same function). One thing to note is that the students were not asked to "evaluate" the integral, that is, put it in closed form (e.g. as a polynomial, in this case). This means that there is a possibility that some students might have evaluated the integral as something different and have assessed *p* and *q* as different for that reason. Of the 100 students, 46 attempted to evaluate the integral, and 29 did so correctly. Unsurprisingly, there is a strong correlation between those who evaluated the integral correctly and those who answered that *p* and *q* are the same function, with 27 out of 29 (93%) who evaluated the integral correctly also claiming that *p* and *q* are the same function, and 8 out of 17 (47%) who evaluated the integral incorrectly claiming that *p* and *q* are not the same function (χ^2=12.4883, p<.05).

The nature of students' incorrect integral evaluations was illuminating and not due to any sort of minor computational errors. In fact, only two students who incorrectly evaluated the integral did so in such a way that it was a function of x (e.g. writing $p(x)=x^3+12$). Instead, 14 out of 17 (82%) included a "+C" in their evaluation of the integral. Students' explanations in 1b have not yet been analyzed to their full potential, but a preliminary reading provides insight to student thinking. Their explanations seem to suggest that some students might have viewed the integral as representing a string of symbols. This is consistent with Musgrave and Thompson's (2014) and Sfard's (1992) findings suggesting that some students think of a function as a string of symbols. To many of the students who evaluated the integral as involving a C (e.g. x^3+C), it would make sense that these students would not think of x^3+C as being the same as x^3-8, as these are different strings of symbols. For example, one student explains "when you derive p(x) it becomes the generalized formula $3x^2+C$. This is not equal to q(x)." Similarly, the students who evaluated the integral correctly tended to find that the resulting string of symbols (x^3-8) was identical to that in the definition of q, and therefore q and p are identical: "once calculated, the integral in p(x) becomes the same expression as q(x)".

There's a sense in which 36 out of 46 gave consistent responses; they either (1) evaluated the integral correctly and wrote that p and q are the same function, or (2) evaluated the integral incorrectly and wrote that p and q are different functions. This is consistent with thinking of a function as a string of symbols; if a student evaluates the integral correctly, then they observe that the resulting string of symbols is the same as x^3-8, and if they evaluate it incorrectly then they observe that the resulting string of symbols is different from x^3-8 (discussed above). The remaining 10 students had mixed responses. Those students' explanations in 1b provide some insight into their understanding of function identity. For example, some students included a +C for the integral yet assessed p and q as the same on the grounds that they share a derivative. Relatedly, some students wrote that p and q are the same function while also stating that they had a different constant. For these students, sameness of derivative was sufficient for sameness of function, and this was reflected in their concept definitions (discussed below).

Coding Question 2 results involved partitioning student answers into "extensional" and "not extensional". "Extensional" includes the characterization of function identity as same graph, same ordered pairs, or same output for every input. Statements such as "g and h are the same when g(x)=h(x)" were not coded as "extensional"; this is because in the absence of a universal quantifier, students could view "g(x)=h(x)" to mean that g(x) and h(x) are identical as equations (strings of symbols) or that g(x) transforms to h(x) under certain rules (Mirin, 2017; Sfard, 1988). Additionally, students might not view g(x) as representing a number or value of a dependent variable and instead view it as a name of a function (Musgrave & Thompson, 2014; Thompson, 1994, 2013b). Thirty-five students' concept definitions were coded as "extensional". Coding the remaining concept definitions is an ongoing project, but it bears mentioning that, consistent with the previous paragraph, 11 students included sameness of derivative in their criteria for function sameness.

I had originally hypothesized that there would be a correlation between students who give extensional function sameness concept definitions and those who answer that p and q are the same function. This is because I expected students with other, non-normative understandings of function identity to claim that p and q are different. This was indeed the case with at least two students, who asserted that p and q differ because one represents an area under a curve, and the other does not. However, a chi square analysis revealed no such correlation. It seems that because p could be expressed in closed form, students' assessment of sameness of p and q was primarily about how they calculated the integral. This allowed for students to assess that p and q are the same on the grounds that they are expressed by the same equation, rather than requiring a robust understanding of function sameness. This resulted in the possibility that students who understand functions as strings of symbols answered that p and q are the same function.

That so many students evaluated the integral with a "+C" is especially revealing. This might suggest that, despite the function notation *p*(x) being used and the quiz explicitly telling them that *p* is a function, these students might not have viewed *p* as a function (perhaps, as one student above put it, "a formula"). This leaves open the possibility that, when these students were asked if *p* and *q* are the same function, they were not viewing *p* as a function at all. This is consistent with the results of the first part of the quiz, in which students appeared to not think of a particular piecewise function as a function (Mirin, 2018).

Conclusions and Future Directions

This preliminary report provides valuable data on students' concept images of function sameness. It is notable that only 61% of these calculus students identified a straightforward instance of the fundamental theorem of calculus as asserting function identity. However, it seems that for several of those students, their assessment was mostly about their calculation of an integral. Perhaps, for the reasons discussed above, we could investigate whether students understand an integral (such as in 1a) to even be a function. It might also be productive to provide a similar function sameness question as in 1a, but instead using the notation f '(x) rather than providing a specific derivative that the student can anti-differentiate procedurally. It might additionally be wise to see how students use and understand the notation "+C".

This study also gives insight into students' concept definitions of function sameness, with 35% providing an extensional (mathematically normative) answer. Interestingly, 11% included sameness of derivative in their criteria for function sameness. I hypothesize that the FTC question might have influenced students' function sameness concept definitions. Future research can address this hypothesis by providing the concept definition question in a different context.

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