LEVEL-ZERO COVARIATIONAL REASONING IN SECONDARY SCHOOL MATHEMATICS

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Based on the results from a research project currently in progress, we outline the hypothesis that exhaustive mathematical study of the behavior of a single variable quantity (E1VQ) constitutes, by itself, a whole cognitive structure (in the Piagetian sense) underlying the development of variational thinking, and that it can and should be fully addressed in secondary education, during and in parallel with the study of representation of numbers and quantities on the number line (a variable quantity – a number axis). This would allow the student to develop a series of conceptual (qualitative) images and quantitative mathematical tools that allow identifying, describing and naming the basic types of variational behavior of a variable quantity.

Keywords: Algebra and algebraic thinking, variable quantities, variational thinking.

Introduction

The ideas addressed in this work are the collateral product of a research project that aims to contribute arguments of a historical, epistemological and cognitive nature in efforts to reconceptualize the teaching of Calculus for non-mathematicians (Jiménez et al., 2020). These ideas point, on the one hand, to the need to promote early development of a variational way of thinking in secondary and high school students, and on the other, to rethink the vision under which the variational approach is embodied in the curriculum of these educational levels. These ideas are presented here as working hypotheses that require a great subsequent effort in educational research, either to corroborate, reject and modify them, or to develop a greater knowledge.

Variational Thinking Conception

The research literature has long documented the difficulties that many students experience in understanding and constructing Cartesian graphs and developing a variational way of thinking (Radford; 2009a, 2009b). In our opinion, one of the epistemological roots of such difficulties consists in the very interpretation of what *variational thinking* is. It is a term whose meaning has not yet been established in a clear and concise way, not to say precise and rigorous, as is typical of scientific work. Different research groups, affiliated with different theoretical approaches, assume their particular interpretation of the meaning of this notion (Vasco, 2010).

In this work, the term ***variational thinking*** refers to the type of mathematical thinking required to understand variation and *change in progress* (Thompson, Ashbrook, & Milner, 2016), and it develops and evolves as the study of such phenomenology occurs. This way of thinking is both *qualitative* (it implies the construction of dynamic images of variation and the reasoning about them, as well as the development of a language that reflects this dynamism) and *quantitative* (it has to do with numerical calculations, techniques and algebraic expressions).

By its nature, the way of thinking that a mathematical study of change in progress requires is a complex entity that we can conceive of as consisting of two components that Thompson and Carlson (2017) have respectively called *variational reasoning* and *covariational reasoning*. These are complementary aspects of the same type of mathematical thinking, which have to do with the mathematical conceptualization of variable quantities.

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Variational Reasoning

Variational reasoning consists of two critical moments. The first is to understand that variable quantities actually *vary*, that is, their numerical values change. Consequently, it is a *dynamic* way of thinking. Thus, this first moment, characterized by the fact of perceiving or conceiving, in a given situation of change, the intervention of one, two, three, four, etc., variable quantities (one of which may be time), and to inquire how they change in order to form dynamic mental images of their ways of changing, to create mathematical tools to represent and quantify such changes, to develop an appropriate language for describing those changes, and much more, is what constitutes the essence of variational reasoning.

There is an important aspect in conceptualizing a variable quantity that has an intuitive connotation. It is the fact that the numerical value a variable quantity takes at each moment is unique: it is not possible for a variable quantity to take two or more different numerical values at the same moment. This essential characteristic of the behavior of variable quantities is known as the *uniqueness principle*.

The second crucial moment in variational reasoning is in some sense an aesthetic appreciation: when a variable quantity changes, it does so smoothly. This "smoothness" is a quality of the processes of change in progress called *continuity*. Time runs smoothly from one instant to the next in an interval; an athlete running moves smoothly from one point to another in the path, and the height of the liquid changes smoothly as a container fills (or empties). The whole process is continuous, since it is continuous at every moment. Usually the process of change-in-progress is continuous, in the sense that it changes smoothly from one state to the next.

The *principle of continuity* is not exclusive to the movement of objects or filling/emptying of containers, but applies to all natural processes. As a process develops, it does not omit any state in its becoming. If the process is in one state at a certain time, and in another state at a different time, then it assumes all states between these two.

In summary, variational reasoning has to do with the mathematical conceptualization of continuous variation of a single variable quantity, this variation having a temporal background.

Covariational Reasoning

The second important stage in variational thinking is to make explicit the fact that in the analyzed situation there are at least two variable quantities present, whose numerical values change simultaneously, and are also related in some way. In this case, the ability to *coordinate* the joint change of these numerical values is crucial for the next level of analysis. Carlson et al. (2002) call this more complex way of thinking as *covariational reasoning*, and characterize it as the set of all "cognitive activities involved in the coordination of two variable quantities while considering how they change in relation to each other". In other words, covariational reasoning about quantities implies the consideration and/or construction of dependency relationships between numerical values of at least two variable quantities that change simultaneously and jointly.

The Early Development Of Covariational Reasoning: Level-Zero

It follows from the previous descriptions that covariational reasoning is cognitively and mathematically much more complex than variational reasoning and that, in order to develop in students the ability to reason covariationally, it is not only desirable but above all it is also ***necessary*** to previously develop their more elemental ability to reason variationally, that is, to reason mathematically about a single variable quantity. Despite this, educational research seems to assume as an unquestionable fact that variational reasoning is simple, unproblematic, natural and spontaneous, and that we should not be concerned with its development. In the overwhelming majority of research work on the subject, the student is involved from the very first moment with mathematical tasks typical of the level of covariational reasoning. The results of our own research point in another direction, suggesting that mastering mathematical work with variable quantities on the number line seems to be a fundamental prerequisite for subsequent work with two number lines in the Cartesian plane. In other words, variational reasoning is related to mathematical work with a single variable quantity on the number line, while elementary covariational reasoning is related to mathematical work with two variable quantities on the Cartesian plane.

Although we know that variable quantities never appear separately in the phenomena of the world in which we live in, isolating and studying them in this way seems to us a justified didactic decision, for several reasons. First, it is a relatively simpler cognitive task, since it does not require the explicit coordination of two variable quantities.

The second argument for first addressing the analysis of a single variable quantity, isolated from the others, is based on the historical development of scientific methodology itself. A methodological strategy to understand complexity is to consider simpler cases of it. The number line is a simpler object than the Cartesian plane.

Our third and main argument is that the basic mathematical ideas required for the coordination of two variable quantities can and should be developed in depth in this simplified, foregoing, hypothetical case (the study of a variable quantity in isolation from the others). Unfortunately, there is no place here to show that this is possible and that it does indeed contain great mathematical richness. In another work (Jiménez et al., work in progress) we argue that, in particular, it is possible to form and develop the mathematical images, tools and terminology necessary to describe and conceptualize the seven basic variational behaviors (uniform growth, accelerated growth, decelerated growth, uniform decrease, accelerated decrease, decelerated decrease, and zero growth or decrease), with a level of complexity similar to that of the case of covariation. Likewise, it is possible to approach some of the advanced ideas of Calculus, such as a first approach to derivatives of higher order and the Fundamental Theorem of Calculus. An outline of such approach is presented in Jiménez et al. (2020).

A cardinal statement of this work is the thesis that *exhaustive mathematical study of the behavior of a single variable quantity* (E1VQ) constitutes, by itself, a whole cognitive structure (in the Piagetian sense, 1968) on which the further development of covariational reasoning relies. A deep understanding of the behavior of a single variable quantity, its description in mathematical terms and its dynamic graphic representation on the number line are relevant and structural for the formation and development of variational thinking, both from a mathematical and a cognitive point of view. In particular, they are fundamental for understanding, interpreting, endowing with meaning and constructing graphs in the Cartesian plane. This is the initial step in the development of variational thought. Given that the process of development of covariational reasoning, according to the terminology proposed by Carlson et al. (2002), has been described in terms of five levels (called L1, L2, L3, L4 and L5), we have decided to name *level-zero* (L0) the corresponding stage of emergence and development of variational reasoning described above.

Related to this, it is sensible to assume that the development in students of a dynamic image of the variable, as well as of a dynamic meaning for it, is possible only relying on a specific mathematical work on the number line, provided that such mathematical work involves the representation of variable quantities and not only of numbers (Jiménez et al., 2020). However, as Thompson and Carlson (2017) rightly pointed out:

> ... there is relatively little research on students' meanings and understandings of number lines. Psychological research in this regard portrays number lines as nebulous objects on which researchers presume that people do informal arithmetic (...), the main interest being by what method people use it to determine sums, products, and so forth. Mathematics education research seems to see the target idea of a number line as being relatively unproblematic and

> focuses on using it as an instructional aid, helping students understand how to locate numbers on it, or using it as a tool in reasoning (...). In both cases, number lines seem to be taken by researchers as lines full of numbers. (Thompson and Carlson, 2017)

The study of the number line is certainly included in the school curriculum, but unfortunately not from the variational point of view. Students learn how to represent numbers by points on the number line, and to associate points with numbers (the coordinates), to identify line segments (intervals), etc. This mathematical work is essentially static in nature. But they never learn how to *represent variable quantities on the number line*, much less how to developn it in dynamic graphic images associated with the different basic variational behaviors, or how to develop and use mathematical tools that allow them to deepen the analysis of the behavior of variable quantities. The number line itself is not a constructed object; it is presented to students as a prefabricated object.

This does not imply that the mathematical work on the number line stipulated by the current curriculum is unnecessary and must be eliminated. On the contrary, it is adequate and necessary, although clearly ***insufficient*** to favor the formation and development of variational reasoning in students. For the latter, it will be necessary to incorporate another type of mathematical work on the number line, of a dynamic nature, related to graphic representation of variable quantities. This implies engaging in a deeper reflection on the construction of the number line itself.

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