DIFFERENCES IN STUDENTS WITH LEARNING DISABILITIES' (LD) UNITS CONSTRUCTION/COORDINATION AND SUBITIZING

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This theoretical discussion provides insight into an intersect of the mathematics education, cognitive psychology, and special education fields. To examine this intersect, the authors focus on how students identified with a learning disability develop actions on material when constructing and coordinating units. This theoretical frame considers results from several case studies in special education and cognitive learning fields, focusing on young students' number development, set in their subitizing activity and units construction/coordination. These results provide context and illustrate critical importance to their actions in light of neural differences and differences in their rate of development for future number and operation construction.

Keywords: Learning Theory; Number Concepts and Operations; Special Education; Cognition

Before children construct arithmetic units, they construct pre-numerical units, evident through a reliance on external representations, such as touching items while counting aloud or flashing four fingers sequentially or with manipulatives in patterned spatial arrangements (MacDonald & Wilkins, 2019). Children construct and reflect on their pre-numerical units to form internal or arithmetic units (Steffe & Cobb, 1988). To internalize units, children would need to step away from a reliance on perceptual material towards material that can stand in for the perceptual units they have constructed. These new pre-numerical units are described as figurative units and evidenced with fingers or counting words. Steffe (2017) estimated that about 40% of first graders do not yet use figurative units when counting and unitizing; this population remains at about 5-8% by third grade. By remaining reliant upon perceptual units, children are not yet able to develop mental operations grounded in their conceptual understandings. These same students are sometimes also identified as having a mathematics learning disability (LD) (Butterworth, 2011). Clements et al. (2013) explain that many children evidence precursors for an LD but are not yet identified, preventing them from receiving targeted mathematics interventions.

To consider how interventions could best be designed, we need to begin leveraging information pertaining to how young children construct pre-numerical units instead of focusing on deficits students with LD evidence (Butterworth, 2011). The purpose of this theoretical commentary is to shift from a deficit model towards a progressive model. In particular, we discuss students' actions and their possible progressions when *subitizing* (a quick apprehension of the numerosity of a small set of items - Kaufman, Lord, Reese, & Volkmann, 1949) and constructing units to determine how students' actions with visual patterns can best support early mathematics development. This, the aim of this theoretical commentary is to *examine aspects of number abstraction processes through students' subitizing activity and/or units construction/coordination.*

Theoretical Framework

To frame this theoretical commentary and consider this aim in the context of special education and mathematics education, we draw broadly from an intersection of cognition and learning and radical constructivist paradigms. In particular, we consider concepts that inform these paradigms: executive functioning (Clements & Sarama, 2019) and units construction and coordination (Norton & Boyce, 2015).

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Executive Functioning

Executive functioning is evidenced through several processes that young children develop throughout their early childhood years (birth to third grade) (Clements & Sarama, 2019). These processes assist children in their ability to self-regulate their learning of mathematics and have been found to positively correlate with children's mathematics achievement (e.g., Best et al., 2011; Clements et al., 2016; Viterbori et al., 2015). In this commentary, we focus on attentional shifting and updating working memory.

*Attentional shiftin*g can explain mathematics strategy development through use of attentional mechanisms (Clements & Sarama, 2019). Attentional shifting is when children are able to shift their attention from perceived material to new perceptual material when developing problem solving strategies. This is evident when young children conceptually subitize (relying on conceptual processes when subitizing). For example, when a four-year-old child is shown five items arranged in a patterned spatial arrangement typical to the face of a die, MacDonald and colleagues (2016; 2019; under review) found children typically subitize two sets of two and one set of one. To segment and unitize, students would need to subitize two and associate this with a verbal word for two. By their attentional shifting between twos and ones, while attending to new information (what warrants attention) and not to other visual material (distractors), children are developing additive strategies (Clements & Sarama, 2019). Many students with LD experience attention differences compared to their peers that contribute to differences when learning mathematics.

When an individual manipulates and maintains information relevant for problem solving, Clements et al. (2013) explain this characterizes students' ability to *update their working memory*. Children engage in this when given multi-step problems, which require their working memory to be engaged and then updated with additional information. For example, when a five-year-old child is shown five items arranged in the same orientation described earlier, and then two additional items are added to this spatial arrangement, this child would be required to hold on to the five items while adding two additional items (possibly combining subitizing and counting). If a child considers the set of five items in isolation to the adding of two items, this child may struggle to construct units for these items, resulting in a counting all strategy. Students with LD struggle with some of these executive functions, pressing them to learn "tricks" grounded in procedural knowledge as they realize their peers are developing more sophisticated strategies for number and operation tasks (Hunt et al., 2019; see also Hunt & Silva, 2020).

Units Construction and Coordination

Units coordination and construction refers to the number of levels and type of units children can construct and bring into a situation (Norton & Boyce, 2015). Prior to units coordination, children use counting to construct *pre-numerical units* in their activity. For instance, children first rely on manipulatives (perceptual units) to construct a pre-numerical unit and determine the total amount through their counting activity. When pressed to step away from the perceptual units, children construct pre-numerical units with finger patterns (figurative units), pointing/tapping (motor units), and/or number words (verbal units) (Steffe & Cobb, 1988). Progressions from perceptual units towards verbal units provide evidence of children transitioning towards internalized actions (imagined external activity). When children interiorize units, these units are considered *arithmetic units* and allows children to operationalize number through their coordination of units (e.g., five is three away from two).

MacDonald and Wilkins (2019) found that one preschool student's subitizing related to her pre-numerical units construction. When developing conceptual processes to assist in her subitizing (e.g., two, two, and one is five; two and three is five), this preschool student constructed perceptual units and then figurative units to evidence her reasoning. Moreover, this student's units were represented with parallel actions (e.g., picking up two manipulatives simultaneously, flashing three fingers). Thus, counting and subitizing activity has been found to inform students of conceptual material that promotes their pre-numerical units construction.

The Intersect of Special Education, Subitizing, and Executive Functioning

Butterworth (2011) found students with LD encode numerosity information differently when subitizing compared to their normal achieving peers. In particular, Butterworth draws from decades of research to explain how young children typically develop *numerosity codes,* where individuals use a particular region of their brain to process sets of items over time and space. Fundamental to number understanding, numerosity codes have been found to evidence deficits in students identified with a LD and explains different types of subitizing activity (Butterworth, 2011). Butterworth explains that this neural difference fundamentally explains why students with LD rely mainly on rudimentary reasoning and strategy development with number (see also Hunt & Silva, 2020).

Hunt et al. (2016) compared findings from clinical interviews involving 21 upper elementary age students with LD with 23 students identified with a mathematics difficulty. Findings evidenced nuances to students with LD's partitioning (partitioning with no regard to equal parts, partitioning with regard to "halves", partitioning with regard to equal parts). When comparing students with LD to students with mathematics difficulties, Hunt et al. (2016) found that 30% of students with LD were able to partition with no regard to equal parts (10%) or with regard to "halves" (20%). Comparatively, students with mathematics difficulties did not rely on such rudimentary partitioning activity. Moreover, 70% of students with LD and 100% of students with mathematics difficulties partitioned with regard to equal parts. These differences suggest some students with LD partition in a very similar way to students experiencing mathematics difficulties, but may be developing their partitioning at a different rate than their peers. These different types of partitioning may also explain working memory differences that students with LD experience when given other tasks that do not provide external representations when working with complicated mathematics concepts. If a student with LD has not yet begun partitioning with equal parts, then solving symbolic fraction tasks may be too much for their working memory to manage. For instance, when solving tasks that only represent fractions as symbols, students may need to consider each symbol as a separate item (e.g., ¾ is considered as a 3 and a 4).

Given these different rates of development, students with LD may evidence seemingly puzzling ways of reasoning that, from a developmental and psychological stance, actually makes sense. For example, Hunt et al. (2019) found that one third grade student, Gina, relied only upon ways of solving number problems using procedures that she could not explain or make sense of. Interestingly, Gina was not perturbed when differences between her procedural number knowledge and physical actions did not align. Yet, when given novel rational number tasks for which she had no procedures for, Gina more readily connected her conceptual knowledge with her actions. Hunt et al. (2019) argued that students with LD are able to develop the same conceptual knowledge as their normal achieving peers, but may be doing so at a different rate. This is important because noticing differences between procedures and physical actions would not be a goal for Gina if procedures were not yet connected in her long-term memory and would make connections back to conceptual understanding difficult.

Conversely, another student, Stu, (Hunt et al., 2016) was also able to anticipate which strategies to use because he was engaged in a platform that supported him to successfully develop equi-partitioning (mental segmenting to form equal parts). In fact, he developed anticipatory types of strategies that allowed him to utilize mental actions so he was not dependent on his physical actions/material to solve problems. Opportunities to develop and abstract the actions that bring about number and rational number affords students with LD opportunities to access tasks that only represent number through symbols.

When considering how these differences evidence themselves in students' subitizing activity, we consider findings from Koontz and Berch (1996) who found elementary age students with LD had significantly slower response times when matching small (two and three) dot arrangements to number words. These findings suggest that young children with LD struggle to update their working memory because items were not processed in a parallel manner and these children struggled to inhibit distracting visual information. When considering this in relation to shifting attention it seemed young children were not yet able to inhibit distracting perceptual material and then shift their attention to primary perceptual material.

Finally, MacDonald et al.'s (under review) findings echo some of these findings, as one first-grade student, Diego, relies mainly on his unitization and iteration actions when solving subitizing and units construction tasks. Findings further indicate that Diego relies heavily on perceptually clustered items when unitizing and not yet able to construct figurative units. These findings also echo Butterworth's (2011) discussion, as he describes an unfinished amount of research examining relationships between students with LD's use of fingers and their numerosity code development.

Conclusion

To date, the research base remains an unfinished work when considering if and/or how students with mathematics difficulties develop separate, or different, understandings of part whole and what features of their diverse cognitive backgrounds (e.g., working memory or attentional processes) might interact with development (e.g., Hunt et al., 2016; Hunt et al., 2019a, 2019b; Hunt & Silva, 2020; Lewis, 2014; Lewis, 2017). In the absence of a convergence of evidence in the research literature, present research efforts document elements of students' diverse cognitive background thought to interplay with children's mathematical learning from an early point in their lives (Compton, Fuchs, Fuchs, Lambert, & Hamlett, 2012). These factors are then used in a predictive sense to explain "learning difference" as variations in certain norms that predict performance over time (Vukovic, 2012).

These findings suggest that students with LD are constructing and coordinating units with partitioning/segmenting activity at a different rate than other students. In fact, we wonder if these different rates of development relate to differences in their development of parallel processing activity, attentional mechanisms, and/or working memory resources. These developmental differences might evidence themselves in their subitizing activity and prevent them access to particular tasks in their early childhood years. To consider these differences more closely, we need to begin adopting new questions and theoretical perspectives, which allow us to work with intersections of mathematics education, special education, and cognitive psychology. Moreover, we need to consider how early actions students construct can explain these differences and inform intervention design that aligns with a wide variety of elementary age students' mathematics development.

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