# PROSPECTIVE TEACHERS' STRATEGIES FOR EVALUATING NON-STANDARD ANGULAR MEASUREMENT TOOLS

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*Angle measure is pervasive within mathematics curricula from elementary school through higher education. Yet, there is evidence that students and teachers alike experience challenges in quantifying angularity. To promote critical thinking about tools for measuring angles in our geometry courses for prospective elementary and middle-grades teachers, we designed non-standard tools and asked prospective teachers whether these tools would be valid for measuring angles. We present these tasks and our analysis of prospective teachers' justifications regarding the validity of these non-standard tools.*

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Measurement is a critical domain of mathematics for students, as well as for prospective teachers (PTs) enrolled in teacher education programs (AMTE, 2017.) Within this domain, angle measure is pervasive within mathematics curricula from elementary school through higher education. Yet, we know from research and teaching that students, as well as prospective and practicing teachers, tend to experience challenges in quantifying angularity (Smith & Barrett, 2017). As mathematics teacher educators, we wanted to help PTs develop productive conceptions of angle measure, so we designed tasks for occasioning conversations about what it means to measure an angle to use in our geometry content courses. In particular, we asked PTs to determine whether several non-standard protractors were valid tools for measuring angles and to provide a justification for their decision. We focused on protractors because, (a) well-prepared beginning teachers are expected to "use measurement tools…[and] are skilled in describing how to select appropriate tools" (AMTE, 2017, p. 78), and (b) U.S. curricula and pedagogy have been critiqued for relying heavily on protractors without sufficiently emphasizing the underlying processes by which a protractor is used to measure angles (Moore, 2012). In this brief report, we present some of these tasks, summarize our analysis of PTs' ways of reasoning about them, and consider implications of these results.

## Tasks and Methods

To promote critical thinking about tools for measuring angles in our geometry courses for PTs, we designed a set of five non-standard tools that might be used for measuring angles. Our intention was to design tasks with the potential for encouraging PTs to think about what marks on a protractor might mean and how a tool for measuring angles might be created in the first place, rather than simply taking conventional protractors and the marks upon them as givens. We refer to the tools we created as *funky protractors* (Hardison & Lee, 2020); these tools are the angular analogue of the "strange" and "broken" ruler tasks others have used to occasion reflection on measuring lengths (e.g., Smith, Males, Dietiker, Lee, & Mosier, 2013; Dietiker, Gonulates, & Smith, 2011). For each funky protractor we designed, we altered one or more features to differentiate a conventional protractor from a funky one (e.g., unequally spaced linear or angular intervals between markings, non-standard shape, etc.). We intentionally designed some funky protractors to be valid tools for measuring angles and some to be problematic for measuring angles (from our perspective). Each funky protractor featured points along the boundary numerically labeled in 10° increments as well as one larger point suggesting a position for placing the vertex of an angle to be measured. Here, we focus on two funky

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protractors, which are both valid tools for measuring angles (Figure 1), and our analysis of prospective teachers' responses to these tasks. In this study we address the following research questions: (1) What decisions do prospective teachers make regarding the validity of potential angular measurement tools? (2) What strategies do PTs use to justify these decisions at the beginning and end of the course?

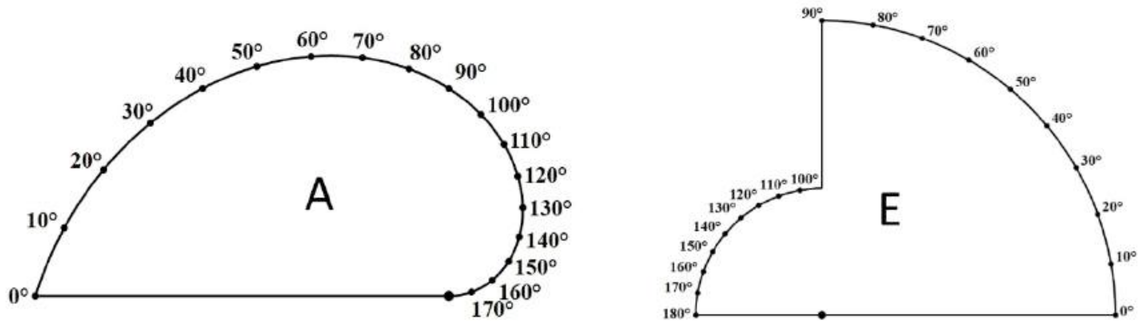


**Figure 1: Two Valid Funky Protractors**

## Participants and Implementation

We implemented the funky protractors tasks with PTs enrolled in three sections of a geometry content course at a large public university; each section was taught by one of the authors. In the second week of the semester, 45 of the PTs evaluated the validity of four funky protractors, including Protractor A (Figure 1), as part of a written homework assignment following a lesson on angle measurement informed by the first author's prior research (Hardison, 2018) and principles of quantitative reasoning (Thompson, 2011). For further details regarding this lesson, see Hardison & Lee (2019). In addition to evaluating the validity of each funky protractor, we asked PTs to explain in writing why each funky protractor was, or was not, a valid tool for measuring angles. After collecting the written responses, we had a whole-class discussion in which PTs discussed their strategies for evaluating the validity of the funky protractors. Fourteen weeks later, 47 of the PTs evaluated the validity of Protractor E (Figure 1) and provided a written justification for their decision as part of their final written exam for the course. We analyzed responses from both the Week 2 homework assignment and final exam.

## Analysis

We coded PTs' written responses for each protractor along two dimensions: validity and justification. We first coded whether PTs determined each protractor to be valid, invalid, or if they failed to make a determination of validity (noncommittal). Then the first author used open coding (Strauss & Corbin, 1998) to establish a set of justification codes for characterizing the rationales PTs wrote to support their decisions. When a stable set of codes was established via iterative analyses, all responses were coded independently by both authors and compared; discrepancies were discussed until consensus was reached. A single validity code was assigned to each response; responses could receive multiple justification codes. Descriptions of the justification codes are provided in the findings section below.

# Findings

Regarding the validity of protractors, on the Week 2 homework assignment, only two of the 45 PTs (4%) identified Protractor A as a valid protractor; the remaining 43 PTs (96%) concluded Protractor A was invalid. In contrast, on the final exam 77% of PTs identified Protractor E as valid, and 21% of PTs deemed Protractor E invalid (see Table 1).

**Table 1: Prevalence of Validity Codes for Protractors A and E**

| Protractor (Week) | # Valid (%) | # Invalid (%) | # Noncommittal (%) |
|---|---|---|---|
| A (Week 2; n=45) | 2 (4%) | 43 (96%) | 0 (0%) |
| E (Week 16; n=47) | 36 (77%) | 10 (21%) | 1 (2%) |

To support their decisions regarding validity, PTs provided a variety of justifications, from which we abstracted six broad categorizations: *attending to measurable attributes, attending to particular angles, attending to shape features, attending to location, using a standard protractor,* and *other*. We assigned *attending to measurable attributes* to justifications indicating attention to extents of successive instantiations of an attribute; for example, justifications reliant upon checking whether the tool indicated a consistent unit received this code. Furthermore, we established subcodes denoting the quantity indicated: *angularity*, *distance between marks*, and *radial distance to marks*. A response received the *angularity* subcode if the PT attended to angular units (e.g., Figure 2, left) and the *distance between marks* subcode if the PT attended to the distance between marks on the boundary (e.g., Figure 2, right). The *radial distance to marks* subcode was assigned when justifications indicated a PT was considering whether marks along the boundary were equidistant from the suggested vertex position. Although this is equivalent to checking whether the boundary formed a circular arc, PTs were not necessarily aware of this. When the particular quantity could not be inferred, we assigned a fourth subcode: *ambiguous quantity*; for example, this subcode was assigned when justifications referred to "even spacing" without further elaboration or annotations. From our perspective, an appropriate justification involves attending to whether the protractor can be used for counting successive angular units of a specified size (i.e., subcode *angularity*).

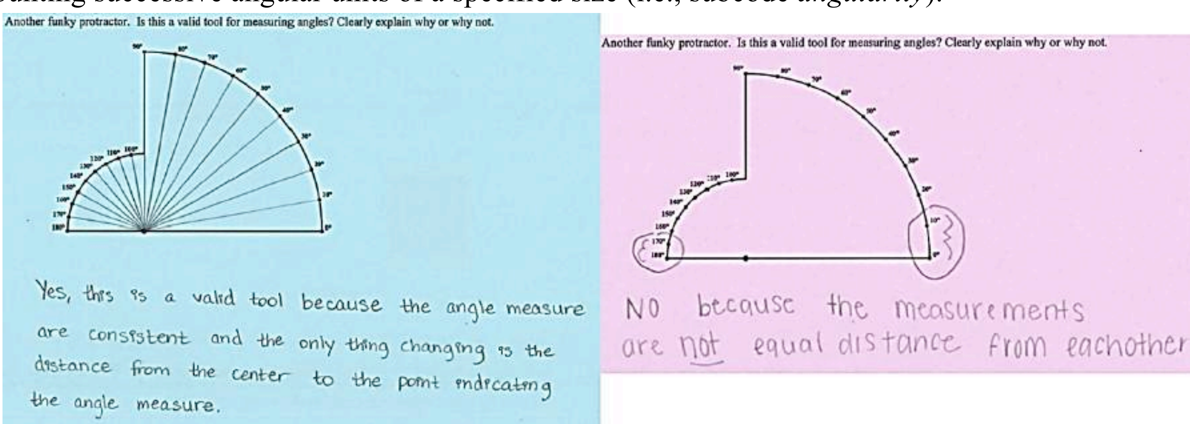


**Figure 2: Justifications Coded as Attending to Measurable Attributes: Angularity (Left) and Distance Between Marks (Right)**

We assigned *attending to particular angles* if a PT's justification was rooted in the measurements of one or more specified angles. For example, some PTs argued that Protractor E was valid because it was appropriate for measuring right and straight angles without indicating whether the protractor would be appropriate for measuring other, arbitrary angles. Justifications were coded as *attending to shape features* when PTs claimed that protractors were valid or invalid based on the shape of the protractor's boundary or the symmetry of the protractor. Justifications referencing position or location were coded as *attending to location*. For example, some PTs argued Protractor E was invalid because the suggested vertex position was not located on the midpoint of the protractor's straight side. *Using a standard protractor* was assigned to responses indicating that a PT physically superimposed a standard protractor atop a funky protractor to evaluate its validity. Finally, justifications that were unclear or did not fit into any of the aforementioned categories were coded as *other*. Codes for PTs' justifications for each protractor are summarized in Table 2; percentages do not sum to 100% because some responses received multiple justification codes.

**Table 2: Prevalence of Justification Codes**

| Protractor (Week) | Measurable Attributes | Particular Angles | Shape Features | Location | Standard Protractor | Other |
|---|---|---|---|---|---|---|
| A (Week 2; n=45) | 24 (53%) | 6 (13%) | 23 (51%) | 4 (9%) | 1 (2%) | 6 (13%) |
| E (Week 16; n=47) | 30 (64%) | 9 (19%) | 2 (4%) | 4 (9%) | 4 (9%) | 11 (23%) |

**Table 3: Prevalence of Measurable Attribute Subcodes**

| Protractor (Week) | Angularity | Between Marks | Radial Distance | Ambiguous Attribute |
|---|---|---|---|---|
| A (Week 2; n=45) | 1 (2%) | 19 (42%) | 1 (2%) | 4 (9%) |
| E (Week 16; n=47) | 11 (23%) | 9 (19%) | 0 (0%) | 10 (21%) |

As shown in Table 2, the majority of PTs (53%) attended to measurable attributes in their justifications for Protractor A. However, as indicated in Table 3, distance between marks along the boundary was the most prevalent attribute indicated in justifications for Protractor A; only one of 45 responses received the *attending to angularity* subcode for Protractor A. The majority of PTs (51%) also attended to shape features, which are irrelevant from our perspective, when evaluating the validity of Protractor A. In contrast, when evaluating Protractor E on the final exam only 4% of PTs attended to shape features. The percentage of justifications coded as attending to measurable attributes increased to 64% for Protractor E with 23% of all responses indicating attending to angularity; additionally, a lower percentage of responses (19%) indicated attending to distance between marks along Protractor E's boundary.

## Concluding Remarks

In closing, we are encouraged by the increase in the percentages of PTs giving appropriate validity determinations and attending to angularity over the course of the study, as well as the decrease in the percentage of PTs attending to shape features; however, differences in the design of Protractors A and E merit cautious interpretation regarding these tasks' impact on PTs' content knowledge. More research is needed to understand how to better support PTs' in developing productive quantifications of angularity. From our perspective, the funky protractors activity and the accompanying classroom discussion afforded opportunities for PTs to think critically about the essential features of valid angular measurement tools. It also afforded opportunities for us as instructors to gain insights into PTs' thinking about measuring angles via the features to which they attended in their justifications.

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