UNDERGRADUATE MATHEMATICS MAJORS' PROBLEM SOLVING AND ARGUMENTATION

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The transition to proof has been a heavily researched area in undergraduate mathematics research. As proof construction involved both formal-rhetorical and problem-solving aspects, this study investigated how undergraduate mathematics majors who recently completed a transition to proofs class engage with two different problems: one with formal rhetorical knowledge and one without. Overall, students exhibited behavior indicating that formal-rhetorical knowledge could continue to act as a barrier to approaching and solving problems. The study highlighted the importance of sense-making strategies and familiarity not only with mathematical content knowledge but also with the logical structure of mathematical arguments.

Keywords: Problem Solving, Reasoning and Proof, University Mathematics, Advanced Mathematical Thinking

Introduction

For much of schooling, mathematics education has often had computation and applications of theorems or formulas at the center. However, the academic discipline of mathematics is more concentrated on generalizations and abstractions; coming up with rigorous and correct proofs of previously unproved statements was the goal for most research mathematicians. This transition to formal proof usually took place during undergraduate studies, with many scholars having examined the cognitive difficulties in learning to do formal mathematical proofs (e.g., Brown, 2007; Moore, 1994; Thoma & Nardi, 2017). As a stark deviation from prior math experiences, this transition is abrupt and difficult (Moore, 1994). Despite the difficulty in learning proofs, research has recognized how proving and problem solving are closely related: It takes problem-solving in order to translate an informal argument into a valid formal proof (Mamona-Downs & Downs, 2009), and the proofs themselves can be divided into formal-rhetorical and problem centered parts (Selden & Selden, 2013). This translation is not an easy task, in that it requires both knowledge and instruction. Students often do not learn the necessary tools explicitly or rigorously until students are asked to construct formal proofs. In this study, we researched undergraduate mathematics students who recently completed their university's proof introductory course. Our research question was: How did these students engage with formal-rhetorical and problem-centered aspects of mathematical tasks?

Framing

The conceptual framework for this study drew from the following main areas: aspects of proof construction (Selden & Selden, 2013; Weber, 2005) and mathematical problem solving (Schoenfeld, 1985) to guide our data collection and analysis.

Aspects of Proof Construction

Though mathematical proofs have been studied for many years prior to their formulation, two considerations of proof construction procedures are particularly useful to this study. One delineates parts of a mathematical proof, whereas the other categorizes different types of proof constructions. Selden and Selden (2013) developed a framework of understanding proofs to be composed of two parts: the formal-rhetorical part and the problem-centered part. Between different statements and

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even different proofs of the same statement, there are different compositions when broken down into these constituent parts. Selden and Selden's idea of these two components of a proof will be the guiding main framework for this study.

Furthermore, Weber (2005) categorized three types of proof construction procedures demonstrated by undergraduate math students – procedural, syntactic, and semantic. Weber argued that learning opportunities are largely dependent on what type of construction students utilized. Together, the formal-rhetorical and problem-centered parts of proof and proof construction procedures provide a framework of understanding students' problem-solving pathways.

Mathematical Problem Solving

Schoenfeld (1985) developed a framework of understanding mathematical problem solving. He delineated four types of knowledge students draw from when engaging in mathematical problem solving: resources, heuristics, control, and beliefs. Others shortly categorized similar ideas in different ways. Considering related research together in a more cognition focused formulation, Schoenfeld (1992) recognized that "there appears to be general agreement on the importance of these five aspects of cognition: the knowledge base, problem solving strategies, monitoring and control, beliefs and affects, and practices" (p. 42). This framework has frequently been applied to college-level mathematics (e.g. Selden & Selden, 2003; Selden & Selden, 2013; Weber, 2005) and is useful for analyzing student problem-solving.

Methods

Ten undergraduate mathematics students, six females and four males, from a Hispanic-serving four-year research university in California participated in this study. All of the students completed a transition to higher mathematics course in the previous quarter. This course, as a prerequisite for many mathematics courses that follow, was described as an introduction to the elements of propositional logic, techniques of mathematical proof, and fundamental mathematical structures, including sets, functions, relations, and other topics.

We conducted and video-recorded task-based, think aloud interviews (Charters, 2003). The participants worked through two problems: a non-routine problem taken from the 2017 American Invitational Mathematics Examination II and a problem that asked students to construct an alternative proof of a well-known result given the argument's outline. The second was developed in consultation with a mathematics professor from the study's university, who taught a number of the participants in the quarter prior to data collection. For the rest of the paper, we will refer to these problems as P1 and P2 respectively. Students chose which problems to attempt first, but they worked through both tasks. The difference between the two tasks was that P1 was a "real-world" problem that needed no formal-rhetorical knowledge, and P2 required substantial formal-rhetorical knowledge (but concluded with the problem-centered part). After the tasks, we collected student work and finished with a quick debrief aimed at understanding the students' feelings about the problems (e.g., their ease and enjoyment) and clarifying the students' strategies. In relation to our research question: the student work informed and documented students' strategies and video data contributed both to our understanding of strategies as well as our understanding of students' feelings of working through the tasks.

To analyze the data, we did an initial round of focused coding (Maxwell, 2005), looking for students' feelings and strategies through the tasks while keeping track whether it related to formal-rhetorical or problem-centered aspects. After the initial coding, we developed another set of codes within strategies: sense-making, argumentation, and generality.

Findings

In general, students exhibited high levels of engagement with and interest in the tasks – every participant requested information about the solutions after the interview. Despite differences in students' feelings about the two tasks, we found similarities about what made the mathematics enjoyable for participants. Regarding strategies, we found that the presence and absence of formal-rhetorical aspects related to how students' sense-making and argumentation.

Students' Feelings: Familiarity, Fun, and Frustration

Familiarity played a role from the start in how students approached the problems. Excluding students whose decisions on which problem to attempt first was semi-arbitrary (i.e., left vs. right, top vs. bottom, etc.), the remaining students were split half and half. Notably, students chose P1 because of an expected enjoyment, whereas others began with P2 because it was more familiar. These reasons were consistent among P1-starters and P2-starters. Isaac, who began with P2, explained that he was, "more comfortable with this kind of thing … there were aspects of everything that [he] could relate to." Conversely, students who started with P1 did so because it seemed more puzzle-like and fun. Henry explained, he "[doesn't] know if [he] considers proof-writing fun," contrasting it with how P1 was "completely uncharted … never seen it before." While the task of constructing a proof felt familiar to students, only some were drawn to this comfort. Although there was certainly a problem-centered aspect to P2, and to proofs generally, the presence of formal-rhetorical aspects seemed to detract from it being seen as fun P1, even when they recognized the proof's puzzle-like aspect afterwards.

There was an interesting interplay between familiarity, fun, and frustration. A common theme among the students was that they enjoyed feeling productive without feeling as though they were doing tedious work. Esther considered P1, "not as much of a math problem." She expressed that sometimes she hard time motivating herself when she could think of how to code a computer program to compute it for her. Furthermore, students had a hard time engaging with the problem-centered part of P2 when they did not know how to approach it. That said, it should be highlighted that frustration was not inherently a bad feeling. Talia found P1 to be no only more frustrating but also more fun. Students found enjoyment when they had tools to approach the task and there was some familiarity, some frustration, and a feeling of progressing in the task without having it feeling immediately solvable by brute force.

Students' Strategies

Generality as a tool and obstacle. As a counting problem, P1 required thinking generally. All of the students recognized this: the students noticed that symmetry and choices were important aspects of the problem. For instance, Madison explained while solving, "If I can figure out how it works for one town, I can just apply that reasoning to other towns." Moreover, all of the students realized that the existence of a five-town loop would satisfy the conditions and attempted to count those configurations. However, though they all recognized it as a sufficient condition, not all of the students investigated whether the existence of a five-town loop was a necessary condition. In fact, Caleb was the only student to substantively explore this.

P2 had two major instances of generality in the desired proof. First, because the proof was a statement holding for every natural number, it sufficed to prove the result using a fixed but arbitrary natural number, maintaining its generality throughout the proof. Second, students had to represent two arbitrary elements of a set when proving that something was closed under multiplication. These uses of generality are routine in proof-writing. The students generally did not have an issue with the navigating the second instance, but there were instances where students reverted to considering all natural numbers simultaneously (as opposed to a fixed arbitrary natural number), which altered the argument's viability. Though students seemed aware of this generality's role, maintaining the correct level generality and applying it to sound argumentation proved to be an obstacle in both problems.

Sense-making to argumentation. Throughout the process, students employed different sense-making strategies. At the lowest level, every participant asked questions, reread the problems, and thought about the tasks to make sure that they understood the problems. However, the process of understanding and solving the problems were manifested differently between P1 and P2.

One of the most frequently used sense-making strategy was thinking of examples. Though this might have been due to the problems' context, every participant's primary approach to P1 entailed drawing examples. These examples then informed their understanding, helping students consider strategies as they constructed systematically. There were fewer instances of student example construction in P2: Isaiah and Gianna listed elements of the sets involved, and Megan drew a number line to explore parts of the problem. That said, the remaining students used symbol manipulation and, at most, tried thinking about the sets in their head. As another sense-making strategy in P1, two students considered a similar, yet simpler, problem (considering a system where there are fewer than the five towns). Additionally, Daniella calculated the total number of road configurations to provide an upper bound to the desired answer. Such a strategy not only provided a means towards an alternate solution path, namely subtracting the invalid configurations from the total to obtain the desired answer, but also provided a measure of reasonableness for future answers. Without this strategy, Henry and Esther got overwhelmed with the sheer number of possibilities and offered infinity as their initial proposed answer.

Largely due to the real-world context of P1, students used intuition to form their strategies. Half of the students clearly made and explored the realization of a working condition. As Talia said, "every town needs a way in and a way out." However, similar to student thinking of five-town loops, there was limited substantive work to show whether this was a sufficient and/or necessary condition. Isaac and Megan both recalled a puzzle that entailed drawing a certain figure without picking up their pencil. Because it seemed related enough, they took it as a potential solution strategy, taking time to realize that they had made an illogical jump to translate P1 into a nonequivalent problem. Conversely, many students explicitly unpacked the logical structure of the statement of P2 and how the argument outlined addressed the logical structure. Some went beyond doing this mentally or verbally, writing the statements translated into symbolic logic. Overall, students generally attended more to sense-making in P1 with sound argumentation taking a backseat, while the opposite was true for P2.

Discussion & Conclusions

The transition to proofs and higher mathematics in general is a complex process that deviates from what many students' previous mathematical experience. While they may not be strangers to mathematical problem solving and non-routine problems, students are asked to integrate formal-rhetorical knowledge when constructing proofs. The presence of this knowledge in problems may overpower the problem-solving that underlie mathematics' appeal. Making explicit connections between formal-rhetorical fluency to sound argumentation and problem-solving generally in instruction may aid understanding and value given to proofs.

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