EXAMINING STUDENTS' REASONING ABOUT MULTIPLE QUANTITIES

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In this report, we discuss five forms of reasoning about multiple quantities that sixth-grade students exhibited as they examined mathematical relationships within the context of science. Specifically, students exhibited forms of sequential, transitive, dependent, and independent multivariational reasoning as well as relational reasoning. We use data from whole-class design experiments with students to illustrate examples of each of these forms of reasoning.

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Variation, Covariation, and Multivariation

Reasoning about variation and covariation has been studied extensively in mathematics education as a way of supporting students' mathematics learning (e.g., Confrey & Smith, 1995; Carlson et al., 2002). More recently, we found that the use of variation and covariational reasoning also supported students' learning of science phenomena, such as the learning of gravity and the greenhouse effect (e.g. Author, 2019; Author, 2020). Science phenomena involve a complex interaction of variables and this provided a constructive space for students to reason about covariation in more complex ways. In these studies, we found that by manipulating the quantities involved in those phenomena using interactive simulations and studying what quantities are changing and how they are changing, sixth grade students exhibited some sophisticated forms of covariational reasoning. Specifically, students coordinated the direction of change of one quantity with the change in another quantity and also identified the bi-direction of change of some of those quantities. Students even discussed inverse relationships, such that as one quantity increases, the other quantity decreases, and predicted the change of one quantity if another is varied multiplicatively. While analyzing our data, we found that students also reasoned about more than two quantities changing simultaneously. Prior research on multivariational reasoning only focused on undergraduate mathematics education (Kuster & Jones, 2019). Therefore, this provided an opportunity to examine students' emerging forms of multivariational reasoning in earlier grades. This effort could eventually respond to Thompson and Carlson's (2017) call for more contributions on defining the covariation construct. Specifically, we aimed to explore: *How do sixth-grade students reason about multiple quantities as they explore complex quantitative relationships in scientific phenomena?*

Theoretical Framework

We use a *quantitative reasoning* lens (Thompson, 1994) to discuss students' forms of reasoning about multiple quantities in the context of science. We use the term *quantity* as one's conceived attribute of an object or phenomenon that is measurable, whether they have carried out that measurement or not (Thompson, 1993; 1994). In this manner, numeric or not, reasoning quantitatively involves analyzing a situation into "a network of quantities and quantitative relationships" (Thompson, 1993, p.1). Accordingly, Kuster and Jones (2019) defined multivariation as a situation with more than two quantities that change in relation to each other. They used this definition to discuss three forms of multivariational reasoning that students exhibited as they explored differential equations: dependent, nested, and independent multivariation. Specifically, they defined *dependent multivariation* as involving at least three quantities that are interdependent with each other, in which a variation in one quantity simultaneously influences the change in other interdependent quantities. They gave the example of reasoning that since P is a function of time, P' is

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also a function of time. They defined *nested multivariation* as involving a network of quantities, where the first quantity is embedded in the second quantity and the change in the second quantity influences the change in the third quantity. For instance, when the differential equation $P' = 2P+2t$ was presented, a student used nested multivariation to explain that a change in t influenced the change in $2t$, then variation in $2t$ changed P'. Finally, they defined *independent multivariation* as involving at least two quantities that are independent to each other and affect the change in another quantity. They gave the example of reasoning that the solution function $P(t)$ is dependent on t, but the rate of change, P', is not influenced by t. Although two independent quantities (t and P') are presented, we would argue that the example does not clearly show independent multivariation because the student does not clearly state that P' influences the function $P(t)$. However, we consider the types that Kuster and Jones presented to be foundational for initiating the discussion around the different forms of multivariational reasoning in the earlier grades.

Forms of Multivariational Reasoning

In this paper, we report on the data from whole-class design experiments (DEs) (Cobb et al., 2003) conducted in three different sixth-grade classrooms, each examining a specific scientific phenomenon: the sea level rise, the water cycle, and the rock cycle. We designed a simulation to dynamically model and study each scientific phenomenon. For example, in the rock cycle simulation students could manipulate a rock's depth and study the changes in its temperature and pressure. We accompanied the simulation exploration with questions that prompted them to reason about those quantitative relationships, such as "How would you describe the relationship between the quantities?" and "How does the change in one quantity affect other quantities?" In the following paragraphs, we discuss five forms of multivariational reasoning that students exhibited (Figure 1) by providing examples of students' episodes from all three DEs.

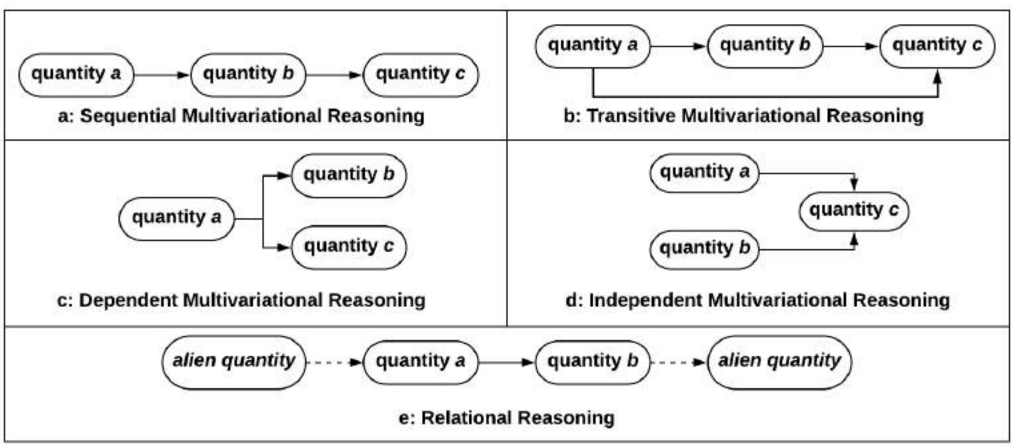

Figure 1: Forms of reasoning about multiple quantities.

Sequential Multivariational Reasoning

In students' articulations, we observed a form of multivariational reasoning that was not discussed in the Kuster and Jones' (2019) study. We refer to sequential multivariational reasoning (Figure 1a) as illustrating sequential changes in quantities, where a change in the first quantity (a) influences a change of the second quantity (b), and a change in the second quantity (b) affects a change in the third quantity (c). While exploring a simulation about sea level rise, students discussed the relationship between the global temperature rise, the height of future sea level, and the total land area. For instance, Myra explained that "The higher the global temperature, the higher the height of the future sea level, and the less the total land area." We interpret her reasoning to illustrate a sequential image of change: that the change in global temperature rise (quantity *a*) impacts the height of future sea level (quantity *b*), and that the change in height of sea level (quantity *b*) affects the change in total land area (quantity *c*).

Transitive Multivariational Reasoning

Our students also exhibited what we would define transitive multivariational reasoning (Figure 1b), a form of reasoning that supports that a change in the first quantity (*a*) leads to a change in the second quantity (*b*), and a change in the second quantity (*b*) in turn changes a third quantity (*c*), then a change in the first quantity (*a*) changes the third quantity (*c*). The difference between transitive reasoning and sequential reasoning is that the transitive reasoning involves the coordination of change in the first quantity (*a*) influencing a change in the third quantity (*c*), which is not illustrated in sequential reasoning. To illustrate this form of reasoning, we provide an example from the water cycle. The water cycle simulation presented a virtual ecosystem, in which students could manipulate the temperatures of air, mountain, land, and lake, and relative humidity and observe the change in the amount of water molecules in every phase of the water cycle. When asked to describe the relationship between evaporation and runoff, Ray stated, "If the rate of evaporation is higher, there could be higher rate of precipitation. If there's a higher rate of precipitation, there could be more runoff. So, the higher rate of evaporation, there can be more runoff." We consider Ray's coordination of the change in three quantities to illustrate transitive multivariational reasoning. In particular, Ray first explained how the change in evaporation (quantity *a*) influences precipitation (quantity *b*), and how the change in precipitation (quantity *b*) influences runoff (quantity *c*). Then he used those two relationships to reason about how a change in evaporation (quantity *a*) causes a change in runoff (quantity *c*).

Dependent Multivariational Reasoning

Our students also illustrated reasoning that we would characterize as a subset of Kuster and Jones' (2019) definition of dependent multivariational reasoning. In contrast to Kuster and Jones' definition in which all three quantities involved are interdependent, the students in our study coordinated a change in an independent quantity *a* which simultaneously affected changes in two dependent quantities *b* and *c*, while quantities *b* and *c* were not related to each other (Figure 1c). For example, when Michael was prompted to describe what he noticed as he explored the rock cycle simulation he stated, "I would say that, the deeper, the deeper you get, the higher the temperature is, and the higher the pressure is." We consider Michael's reasoning about the relationship of depth with the temperature and pressure to be dependent multivariational reasoning. Michael's language "the deeper" and "the higher" also shows an understanding of simultaneous change between the two dependent quantities (temperature and pressure) as influenced by one independent quantity (depth).

Independent Multivariational Reasoning

Our students exhibited independent multivariational reasoning, (Figure 1d), similar to Kuster and Jones' (2019) definition of coordinating a change in two independent quantities (quantities *a* and *b*) influencing the same dependent quantity (quantity *c*). For example, when Chloe and Justin were asked to use the water cycle simulation to release snow by manipulating only the air temperature and the land temperature, they reasoned that "We need both of them to be cold." Chloe explained that "if you just move for air temperature, it only snows a little bit, but if you put it with a land temperature, it starts to accumulate in the ground and it produces more." Chloe illustrated an example of independent multivariational reasoning as she coordinated the change of land temperature and air temperature as unrelated independent quantities with the change in snow as the dependent quantity.

Relational Reasoning

In addition to the above four types of multivariational reasoning, we also noticed instances where students related their explorations with quantities that were not part of the specific study. We refer to relational reasoning (Figure 1e) as the form of reasoning that connects the relationship of two quantities with a third quantity that students bring in from their prior experiences (what we refer to as an *alien* quantity). Relational reasoning can be expressed together with other forms, such as sequential multivariational reasoning. For instance, while exploring the water cycle simulation, we asked students to explain the model. Lorna connected the relationship between the amount of precipitation, runoff, and infiltration with the quantity of water that would go into the aquifers, which was not identified in the simulation or module. Lorna reasoned that "the more rain there is, there's more runoff. And the more runoff, the more water is going to go into the aquifers." Lorna first reasoned about the change in the quantity of rain with change the quantity of runoff. Then she coordinated the change in runoff with the amount of infiltrated water in the aquifer, an alien quantity to the simulation.

Conclusions

In 2017, Thompson and Carlson argued that while there are a wealth of studies employing variation and covariation as a framework for their investigations, these "do not contribute directly to defining the construct" (p. 427). Investigating how students may reason about more than two quantities makes a contribution to this call. The Kuster and Jones' (2019) study initiated a discussion about how we can define students' forms of multivariational reasoning. Our study built on their work to examine how students as young as sixth grade could reason about multiple quantities. By exploring the sequential and simultaneous variation of quantities involved in the water cycle, rock cycle, and sea level rise phenomena, students exhibited five different forms of reasoning about multiple quantities, namely sequential, transitive, dependent, and independent multivariational reasoning as well as relational reasoning.

The retrospective analysis showed that it was the students' interaction with the simulations and the probing questioning that provided a constructive space for them to study the variation in multiple quantities and reason multivariationally. Our initial goal in the study was to engineer opportunities for students to reason covariationally, therefore our tasks and questioning were restricted to only a few prompts to connect multiple quantities. In the next iteration of our design, we plan to engineer more opportunities of this type of reasoning. Through this process, we can examine the progression from covariational to multivariational reasoning and the tasks, tools, and questioning that assist students in exhibiting each specific form of reasoning about multiple quantities.

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