PROSPECTIVE HIGH SCHOOL MATHEMATICS TEACHERS' USES OF DIAGRAMS AND GEOMETRIC TRANSFORMATIONS WHILE REASONING ABOUT GEOMETRIC PROOF TASKS

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The purpose of the study was to examine prospective teachers' uses of diagrams and approaches to congruence while solving proof tasks. Eight prospective high school mathematics teachers were given two proof tasks to solve at the beginning and end of a mathematics education course. Analysis revealed that at the beginning of the course preservice teachers' approached congruence proofs using a perceptual or correspondence approach and interacted and used a descriptive mode of interaction with diagrams. At the end, their approaches to congruence included more instances of transformations and measures and their interactions with diagrams included fewer uses of the descriptive mode and more instances of representational and functional modes.

Keywords: Geometry and Geometrical and Spatial Thinking; Reasoning and Proof; Representations and Visualization

Introduction and Related Literature

The study of geometry in high school is often students' first experiences with conjecturing, justification, and formal proof. Most mathematics standards recommend that students be familiar with different approaches to proof that include synthetic, analytic, and transformational methods (Coxford, 1991; National Governors Association Center for Best Practices & Council of Chief State School Officers, 2010). Yet many teachers have not had experiences using geometric transformations to write formal proofs. There is also research that suggests students' and teachers' interactions with diagrams can support their conjecturing and proving activities (Herbst, 2004; Gonzalez & Herbst, 2009; Chen & Herbst, 2013).

There is significant research related to students' and teachers' abilities to write formal proofs. In a recent research synthesis, Stylianides, Stylianides, and Weber (2017) identified three perspectives on proof abstracted from the literature: "proving as problem solving, proving as convincing, and proving as a socially embedded activity" (p. 239). We adapted a proving as problem solving perspective in which participants were presented four proof tasks to solve in a task-based interview setting. Within this perspective, Selden and Selden (2013) make distinctions between the formal-rhetorical part of a proof and the problem-centered part of the proof. The former focuses on the logical sequencing of steps when writing a formal proof while the latter refers to the creative problem solving that is involved in considering how one might go about proving a conjecture. The problem-centered part of proving is similar to the identification of a proof plan (Melis & Leron, 1999) or proof idea (Reiss, Heinze, Renkl, Grob, 2008) that occurs prior to the writing of a formal proof. It is within this area that we focus our analysis on describing how preservice teachers interact with diagrams while constructing proofs involving congruence.

Gonzalez and Herbst (2009) investigated high school students' conceptions of congruence and then identified perceptual (PERC), correspondence (CORR), transformational (TRANS), and measure preserving (MeaP) conceptions of congruence. The perceptual conception is one that relies on visual information provided in a diagram to determine if two objects appear congruent. The correspondence conception is one in which two objects are congruent if corresponding sides and angles are congruent. The transformational conception uses properties of geometric transformations to map one geometric object to another. The measuring conception relies on measures of objects to determine if

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they are congruent. The tasks selected for the current study could be approached by students holding any of these four different conceptions of congruence.

Conceptual Framework

Building on the work of Duval (1995), Herbst (2004) proposed four modes of students' interaction with diagrams as empirical (EMP), representational (REPR), descriptive (DESC), and generative. With empirical interactions, the actor has proximal, physical experiences with diagrams. The actor's operations on diagrams (measuring, looking, drawing) is limited to actual properties of the physical drawing. This identifies the diagram as an object; that is, a diagram is taken as a figure without semiotic mediation (Chen & Herbst, 2013). In the representational mode, the actor uses distal physical experiences to make depictions about the diagram and the diagram is seen as a sign of the object. Herbst (2004) also suggests two other modes of interactions, descriptive and generative, to characterize the role of diagrams in the process of proving. In the descriptive mode the actor sets up a distal relationship with a diagram while making statements that could be read off the diagram. Also students use visual perception when they are doing proofs and verify this perception by additional symbols like hash marks or arcs. This mode is a hybrid mode that students use both visual perception to make conjectures like the empirical mode and also see diagrams as symbols to justifying their statements like the representational mode when proving (Chen & Herbst, 2013). Conversely, within the generative mode, students make sensible changes that are not originally given and make "reasoned conjectures" in predicting and making hypotheses about the figure. Students interact in proximal relationship and work generatively with diagrams by using definitions and properties of the geometric objects as well as making changes. Gonzalez and Herbst (2009) proposed the functional mode (FUNC) of interaction to define students' interactions with dynamic geometry diagrams. They describe how students relate outputs and inputs when they use the dragging feature of the dynamic geometry software. Within this mode the combination of dragging and measuring provides students opportunities to explore relationships. Students may also check invariants when making changes to the diagram by dragging and set up the same relation between several diagrams. The purpose of this study was to examine preservice teachers' (PT) interactions with diagrams as they solved proof tasks that were amenable to synthetic or transformational approaches.

Context and Methods

The current study took place at a large public university. Eight preservice (PT) high school mathematics teachers (four males and four females, identified as S1-S8) enrolled in a senior level mathematics education course agreed to participate. Approximately three weeks of the course were devoted to the study of transformations, congruence, and similarity. An emphasis on proof and justification was included throughout the course which addressed number (real, complex), rates of change, functions (linear, exponential, logarithmic), and statistics. The participants were required to solve three tasks at the beginning and four tasks (three were the same) at the end of the semester. At the beginning, PTs were provided iPads with the ShowMe app (interactive whiteboard app) and asked to record themselves solving the tasks. At the end, PTs were invited to participate in task-based interviews; they were provided with the same materials and technology they had used in class. For this paper, analysis of the first two tasks is provided. These tasks were selected and adapted from high school mathematics curriculum and prior research that emphasized transformational and synthetic approaches to proof. Task 1 was adapted from the Mathematics Vision Project Secondary II Curriculum (Module 5, page 16, https://www.mathematicsvisionproject.org/secondary-mathematics-ii.html). Task 2 was modified from professional development materials created by Jim King that were used to prepare teachers to teach congruence using a transformation approach (Figure 1).

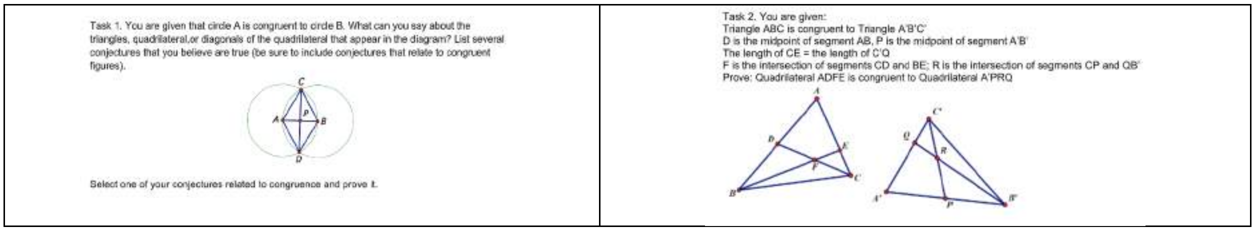

Figure 1. The first two geometry tasks used in the study

For each of the eight PTs, video recordings of their work on the first two tasks from the beginning and end were reviewed and coded to characterize their approach to proving congruence and coded to identify how they interacted with diagrams. The data were analysed by the researchers independently in line with the theoretical framework. Afterwards researchers discussed and agreed on the codes for trustworthiness and consistency.

Results

The most common conception of congruence involved a combination of the CORR with PERC based reasoning. At the beginning, only one PT (S4) used a TRANS approach on the first task. This PT and one other (S6), changed the placement of the second triangle in the second task for ease of determining which sides and angles corresponded to one another. Although this repositioning involved a rotation, this strategy was not used to justify why the quadrilaterals were congruent and thus not coded as a TRANS approach. At the end three PTs (S3, S4, S8) used a TRANS approach. Also only two PTs chose to use dynamic geometry in solving the first task (S1, S8). When examining PTs' interactions with diagrams, we note that there were 11 instances of the DESC, two instances of the REPR, and three instances of the GENE at the beginning. At the end, there was a greater variety in the types of interactions with diagrams. There were seven instances of the DESC, three instances of the REPR, three instances of GENE, and two instances of FUNC mode.

On the first task, all teachers at the beginning made conjectures about the two equilateral triangles and the four congruent right triangles created by the circles. Three of the teachers (S1, S4, S8) made conjectures about the quadrilateral and among these only S4 used reflections in his proof. Most of the PTs noticed that the sides of the quadrilateral are radii of the two circles and used that information to prove triangles congruent. The PTs who made conjectures about the quadrilateral proved it was a parallelogram (S1) or a rhombus (S4, S8). Almost all PTs (except S4) built their conjectures based on the PERC. Even if teachers approached the first task by PERC, they also used a CORR (coded as PERC-CORR). Only S4 utilized the TRANS during the reasoning process at the beginning. None of the teachers used dynamic geometry at the beginning. At the beginning teachers generally interacted with figures DESC, but there are GENE (S2, S4) and a REPR instances (S8). At the end S1 and S8 tried to prove their conjectures for Task 1 using dynamic geometry. Especially, S1 measured all the line segments and used the drag test to justify his/her conjectures (congruent triangles) and S8 used reflections as well as dragging. Also there was one MeaP, three instances of PERC, two instances of CORR, and two instances of TRANS conceptions. From the point of interaction with diagrams, PTs' interactions have varied at the end as three instances involve a DESC, two instances of REPR, one instance of GENE, and two instances of FUNC mode of interaction with diagrams.

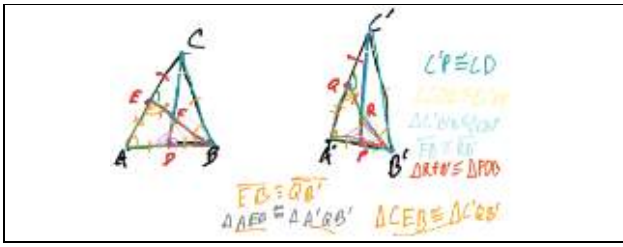

Figure 2: Example of correspondence approach

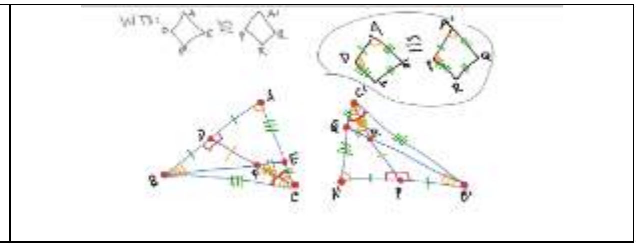

Figure 3: Example of generative mode

On the second task, at the beginning most teachers (S1, S2, S5, S6, S8) used a CORR approach combined with the PERC. They often started the task by marking the given information and using the fact that the two triangles are congruent to identify corresponding parts. In the process of constructing their proof they often made inferences about congruent objects from the diagram using visual perception. At the end, there were no instances of a completely PERC and three instances of the use of TRANS (S3, S4, S8), one of which was combined with a CORR. PTs' descriptions of each of the TRANS described the transformation (e.g., rotation, translation) and stated where points would be mapped, but did not specify a center and angle of rotation or a translation vector (Figure 4). Four PTs used a combination of perceptual and correspondence approaches and one PT decided to skip this question.

Discussion

When looking across the eight participants and two tasks implemented before and after the course we observed that there were no instances of EMP interactions. This is not surprising since most observations of empirical interactions with diagrams occur before high school (Herbst, 2004). Although there was no change in the number of GENE, only one participant was the same and two new participants used this mode. The number of REPR modes increased by one and the number of DESC decreased from 11 to 7. The appearance of the FUNC mode was identified in participants who used dynamic geometry. Analysis of the conceptions of congruence across participants shows more variation in the ways PTs reason. While at the beginning participants wrote proofs that primarily used PERC and CORR approaches, at the end participants used MeaP when using dynamic geometry and used transformations more often.

The identification of the interaction between PERC and CORR approaches to congruence was useful to the researchers in describing how PTs engaged in proof problems using an approach with which they were familiar (correspondence), but when unsure about how to continue made inferences from the diagram based on visual information to proceed with the proof. Their proof idea, proving two figures congruent using a CORR approach, was correct, but it was in the details of formalizing that idea that they encountered challenges. While Selden and Selden (2013) make distinctions between the formal-rhetorical part of a proof and the problem-centered part of the proof the challenge experienced by many of our participants seemed to lie somewhere between these two activities. While in many cases they understood how to go about solving the proof problem, it was in the details of logically moving from one step to the next where they encountered challenges. This aspect of proof writing might be worth examining in future research.

Acknowledgments

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