MAKING ADDITION VISUAL: SUBITIZING AND SCAFFOLDING

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In this qualitative empirical study, we discuss new perspectives on how the teaching of addition can be made visual for young learners. Our research is framed by scaffolding (specifically the development of conceptual discourse through representational tools) and subitizing. Subitizing is generally described as seeing how many suddenly without counting each individual item and representation tools are resources that support students and teachers to discuss and reflect on mathematical ideas. We describe one of many short Grade 2 classroom episodes that occurred weekly during an entire school year in Canada. Episodes were centered around the use of small round manipulatives that were arranged based on subitizing literature. We make initial claims about making the teaching of addition visual through student subitizing abilities, correctly solving an arithmetic problem and student explanations and responses to what they saw and did.

Keywords: Number Concepts and Operations, Elementary School Education.

Objective

By examining a short classroom episode on addition through the lenses of scaffolding and subitizing, this study aligns with the PME-NA goal of deepening and understanding the psychological aspects of the teaching and learning of mathematics. Using subitizing, a construct that is firmly rooted in psychology (Kaufman et al., 1949; Revkin, et al., 2008), leads to new perspectives and pathways to discuss the issues of teaching number concepts and operations. This qualitative study pairs mathematics education literature (e.g., scaffolding) with psychological concepts (e.g., subitizing) with the aim of examining the question: How can the teaching of addition be grounded in visualization through capitalizing on subitizing?

The difficulties and issues young students face when learning about how to solve basic arithmetic problems are well documented in mathematics education research (e.g., Boaler, 2015; Baroody et al., 2009; Jordan & Montani, 1997) and is even apparent in media pieces with public calls for 'back to the basics' (Rushowy, 2019). Boaler (2015) states, "[w]hen students focus on memorizing…they often memorize facts without number sense, which means they are very limited in what they can do and are prone to making errors" (p. 2). Similarily, Kamii and Domenick (1998) suggest that when young students are pushed to memorizing algorithms too soon (such as the traditional addition algorithm where students stack the numbers and "carry"), the algorithms "unteach" place value which in turn hinders the acquisition of number sense.

'How to' manuals and daily teaching activities have appeared and have great potential to address issues of basic arithmetic. The two we identify as having the most potential and influence are 'number talks' and 'making thinking visible'. These activities and manuals could benefit from conceptual framing and infusing/weaving of psychological research with explicit connections to teaching. Number talks are brief daily talks where students talk about their strategies to mentally solve computational questions (Humphreys & Parker, 2015). Number talks (Parrish, 2010) align with research findings that suggest that students learn basic arithmetic gradually over time (Bruce & Chang, 2013). 'Making thinking visible' for the purpose of enhancing teaching can be found in the form of books for teachers (e.g., Hull, Balka, D. S., & Miles, 2011). Ritchart, Church and Morrison (2011) explain how active use of knowledge (including retention and understanding) is achieved through learning experiences that require learners to think about and with what they are learning.

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Understanding and explicating student thinking is a difficult task (Leatham et al., 2015) because thinking is largely invisible and often conceived as an internal process (Ritchhart, Church, Morrison, 2011).

Here, ideas from 'number talks' and 'visible thinking' inspired us to develop short classroom episodes on addition and subtraction where small round manipulative were spatially arranged based on recommendations in subitizing research. In essence, these lessons aimed to make individual student's addition and subtraction solution strategies visible so that they could be discussed.

Conceptual framing

Subitizing is generally described as "instantly seeing how many" (Clements, 1999, p. 400). Clements (1999) categorized subitizing into two types: perceptual and conceptual. Perceptual subitizing is "[r]ecognizing a number without consciously using other mental or mathematical processes and then naming it" (Clements, 1999, p. 401). Whereas, conceptual subitizing applies the perceptual process repeatedly and quickly uniting those numbers. For example, a child can perceptual subitize "4" by simply recognizing it and naming it and conceptually subitize 4 by recognizing 4 consists of 2 groups of 2. It is important to note that the way objects are spatially arranged can impact the ease at which a student subitizes (Clements, 1999). Indeed, studies (e.g., Mandler & Shebo, 1982) about subitizing have concluded that students make less mistakes (i.e., find it easier) when dots appear as they do on dominoes—e.g. 10 as two sets of five as you would see a five on a dice face.

Subitizing is deeply linked to visualization and images, and it has clearly been suggested that subitizing should be capitalized on for the learning and teaching of addition (Clements, et al., 2019). Clements (1999), while making these recommendations, draws on the work of Markovits and Hershkowitz (1997) to go as far as saying that "[c]onceptual subitizing is a component of visualization in all its forms…and [c]hildren refer to mental images when they discuss their strategies" (p. 403). Noteworthy is that a literature search on subitizing and number operations reveals many pieces from psychology and points to a scarcity of classroom-based research in how subitizing can be/has been used in the service of teaching addition.

The wide use of scaffolding in math education is apparent in Bakker, Smit, and Wegerif's (2015) literature review. In 2006, Anghileri related scaffolding specifically to math contexts based on previous research on scaffolding outside of math (e.g., Wood et al., 1976). She put forward three levels of scaffolding with the aim to provide language that can be used to describe actual acts of teaching in mathematics classrooms. Scholars (e.g., Bakker, Smit, and Wegerif, 2015), have used Anghileri's three leveled framework to analyze data from math classrooms.

Level 3: *developing conceptual thinking* comprises of two subcategories: making connections and developing representational tools. This level is "less commonly found but identified as the most effective interactions" (p. 47). Level 3 scaffolds are most relevant to this study because of representational tools. Representational tools support students and teachers in building conceptual discourse, as they "constitute as a resource that students can use to express, communicate, and reflect on their mathematical activity" (p. 48). Anghileri (2006) explains that representational tools can provide "powerful visual imagery" (p. 47) and are important because "[m]uch of mathematical learning relates to the interpretation and use of systems of images, words, and symbols that are integral to mathematical reasoning" (p.47). The most common form of representational tools take the form of symbolic records of students' ideas that occur through teachers noting students' interpretations and solution strategies. Representational tools can take other forms such as graphs, and in Dove and Hollenbrands' (2015) study which examined scaffolds provided by high school geometry teachers, Geometer's Sketchpad was considered a representational tool.

In terms of most effective teaching and Level 3 scaffolds, Anghleri (2006) gives a sense that mathematics classrooms should go beyond the individual and there should be evidence of "shared", "togetherness", "communal" and "cooperation". This is apparent when she describes "real mathematics learning in the classroom" in terms of "struggle for shared meaning… a process of cooperatively figuring things out determines what can be said and understood by both teacher and students" (p. 46). She explains that with Level 3 scaffolding "understanding comes to be shared as the individuals engage in the communal act of making mathematical meanings" (p.49). Meaning that effective teaching (or *developing conceptual thinking)* can be evidenced through the development and the use of representational tools that are communal.

Methods

This study took place in a Grade 2 class, in which, researchers and the classroom teacher co-planned short (typically 7-8 minute) weekly lessons/episodes, throughout the 2018-2019 school year. There were approximately 20 students involved in each episode which took place in a predominantly English-speaking public school in a highly populated Canadian city. One of the researchers (not the assigned teacher) acted as the lead teacher during the classroom episodes.

During the lessons, the study participants were seated on a carpet together in front of the screen. Students were given an arithmetic problem (18+12) and asked to think about how they would solve it. Small circular objects (flat circular candies that are ¾ cm in diameter) were laid out and projected on a screen to represent numbers in questions. Numbers were often colour coded (i.e., for 18+12; 18 was represented by blue objects and 12 by green objects, Figure 1).

Eight of the 42 collected episodes were analyzed for this report. One episode was chosen because our conceptual framework points to the development of representational tools being communal acts that involves 'shared' and 'together'. Hence, we looked for what we call 'chorus' in the data. These are incidents in the videos where students joined in 'chorus' speaking with the teacher or in response to teacher questions about what they saw. It is difficult to identify, from the audio of the video recorded data, how many students form chorus but it is clear that it is much more than 10 students. Data was analyzed using Powell et al.'s (2003) model for studying the development of learners' mathematical ideas and reasoning using videotape data.

Results (Episode of 18+12)

The episode begins with the teacher directing students' attention to a pile of 18 blue candies and a pile of 12 green candies (with no structured arrangement) and asking students how many blue candies there are. A student comes to the projector and begins to move the candies one by one to create two strings of 3 candies each. The teacher interrupts the student and says: "Can I offer you an idea? I would really like 5s" and the teacher arranges five of the blue candies as you would see on the face of a dice. Before the student begins to re-arrange the objects, the teacher addresses the entire class with: "Does everyone agree this is five?" as she circles the five candies she re-arranged with her finger. A chorus of students calls out "yes". Subitizing is confirmed with immediate student chorus confirmation to the question "can you see it right away?".

The student rearranges the rest of the blue candies and declares 18. The teacher verbally repeats the number 18 and asks the student "How do you know that?". The student explains he (re)composed the larger number 18 by saying "five, ten, fifteen, eighteen" while simultaneously pointing at the manipulatives. The teacher says: "OK. Five, ten, fifteen, 16, 17, 18". There is a chorus of students that join for "five, ten, fifteen" only the teacher says "16", more students join for 17, and then there is a chorus for 18. Another student moves up to the projector and rearranged the green candies in a similar way. The student states: "five, ten, eleven, twelve" providing evidence of conceptual subitizing. The students are then told the goal of the teaching episode is to figure out how many candies are on the projector (i.e., to solve 18 + 12) and figure out as many strategies as they can.

The teacher invites another student to the projector to use counting on as a strategy to solve 18 +12. The student starts by recognizing the 18 blue candies. Then the teacher and student count one-by-one together using their hands and fingers “19, 20, 21, …” until 30 (note that the manipulatives are not used). The teacher then prompts for connection making by pointing to the candies individually as she says “19, 20, 21, …” until 30. There is no audible chorus with her as she spoke and she asks how many students used the strategy of counting on. Two other students lift their hands to indicate they used a solution strategy of counting on by 1s.

The teacher asked, “who used a different strategy?” A student offers a strategy that counts by 5s and 10s. The student starts by pointing at two blue groups of 5s that are already formed and saying “five, ten, fifteen, twenty, twenty-five [pause] then I moved these two and put them together” as she moved the two green candies to be with the three blue candies to form a five as you would see on the face of a dice (Figure 1).

Figure 1: 18 + 12

Discussion and/or Conclusions

We conclude that addition was made visual, as teaching was grounded in visualization. Given the issues that students experience with addition, this study should be of great interest, as we offer a different way to teach addition that follows underlying principles of teacher resources (‘Number Talks’ and ‘making thinking visible’) but extends practical suggestions by infusing math education research and psychology research. In essence, we have shown how the teaching of addition can be made visual through using small objects that are spatial arranged in specific ways that are inline with research on subitizing.

Our results evidence teaching grounded in visualization through conceptual subitizing (“a component of visualization in all its forms”) and the development and use of a representational tool that is communal. Students used conceptual subitizing to identify 12 and 18 items. There are indications of a representational tool that is communal when students respond in chorus and when solving 18+12. Spatially arranging small round manipulatives in a very specific way provoked one student to describe how she solved 18+12 by composing the last group of 5 (conceptually subitizing 5) by composing 3 blue and 2 green candies.

Although it has been recommended that subitizing can and should be used to support students in arithmetic, there is a scarcity of studies that respond to these recommendations and calls by enacting and researching them in actual mathematics classrooms. Significantly, through conducting classroom based research and analyzing our data through scaffolding and subitizing, we highlight how number talks that focus on making thinking visible by capitalizing on subitizing can be used to make the teaching of addition grounded in visualization.

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