# STAGES IN USING PROOF TECHNIQUES: STUDENT DEVELOPMENT IN THE TRANSITION TO PROOF

## ETAPAS EN EL USO DE TÉCNICAS DE PRUEBA: DESARROLLO DE LOS ESTUDIANTES EN LA TRANSICIÓN A LA PRUEBA

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*The transition to learning how to prove is difficult for undergraduates. We are aware of the many varied struggles students have, but we know less about their development as they are learning. This is vital as development is more than just an accumulation of competencies. To examine these developments, a series of four task-based interviews across a semester were conducted with (N=11) undergraduate students enrolled in a transition to proof course. Video of students constructing proofs was analyzed qualitatively; changes in how students chose what proof technique to use were common. Stages in students' rationales are illustrated using two students as cases. The results show students' decision-making in starting a proof and remind us that such judgement takes time to grow. Instructors and curriculum developers may use these results in designing tasks and supports for the transition-to-proof.*

Keywords: Reasoning and Proof, Post-Secondary Education

The transition-to-proof is difficult for undergraduate students (Moore, 1994; Selden & Selden, 1987). Students struggle with learning how to prove (Iannone & Inglis, 2010; Selden & Selden, 2013). The transition-to-proof is a shift in the "game" of mathematics, from answering "exercises" that are largely procedural (Schoenfeld, 1992) to now writing arguments and justifications. Researchers have identified the types of errors students make (Selden & Selden, 1987) and their struggles (Selden & Selden, 2003): use of examples, notation and symbols, quantifiers, and general logic (Epp, 2003; Selden & Selden, 1987). Students also struggle with larger issues, such as providing empirical rather than deductive arguments (Harel & Sowder, 2007) and having difficulty writing formal arguments (Alcock & Weber, 2010). Another strand of research has focused on students' strategies and approaches to the proving process (Karunakaran, 2014; Savic, 2012).

We know then students' struggles and strategies while proving at singular points in time, but few have looked at how students develop, at how their strategies change over the course of the learning process. Development is not necessarily about accumulating competencies: "For some psychologists, development is reduced to a series of specific learned items, and development is thus the sum, the culmination of this series of specific items. I think this is an atomistic view which deforms the real state of things" (Piaget, 1964, p. 38). Thinking about proving as the sum of skills and assessing whether or not students have those skills is not enough for us to understand students' learning process. We do not yet know *how* students put the pieces together while they are learning how to prove nor the *order* in which they occur. We lack models of students' cognitive development in this domain.

In response to this gap, the research question guiding this work is: How does undergraduate students' proving develop over the duration of a transition to proof class? The purpose of this study is to understand how students come to learn how to prove. In this paper, I examine one prevalent development that occurred and illustrate it through two participants.

## Conceptual Framework

### Proving as Problem Solving

While there are multiple ways to think about proving as an activity, I take the conceptualization of proving as a form of problem solving (Stylianides et al., 2017). I further take problem solving to be the activity a person engages in when stuck, reaching an impasse (Savic, 2012). Under this definition, a task can elicit problem solving in one student but not another, depending on whether or not they become stuck at any point. There are a lack of robust frameworks for characterizing a student's proving (Savic, 2012), but by considering proving as problem solving, we can look to work on problem solving. I focused on the components of strategies (heuristics) and monitoring and judgement of problem solving (Schoenfeld, 1985b; 1992). Moreover, the focus here was on proving as a *process* (Karunakaran, 2014), rather than on the product, the correct proof.

Development at its most base level may refer to change over time. Development does not happen in a vacuum; it is undoubtedly informed by instruction. A common way to consider development is in terms of stages, in which a person passes through each stage on their way to full mastery (e.g. Lo, Grant & Flowers, 2008; van Hiele, 1959). I conceptualize development simply as taking a "snapshot" - a characterization of some construct at a point in time - and looking across these at multiple timestamps for change (Figure 1).

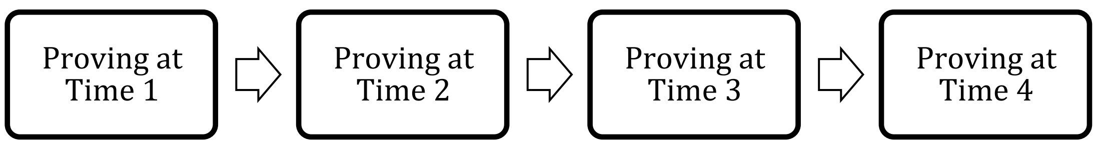


**Figure 1: Conceptualization of development, by capturing snapshots of student's proving and comparing over time.**

The purpose in taking this simple view of development is to provide as much description as possible and look for natural change, which may then inform the creation of potential frameworks and models for how students develop in a transition to proof course.

## Method

A series of four semi-structured interviews were conducted with N=11 undergraduate students in a transition to proof mathematics course at a large Midwestern university. Their ages were 18 and up. This transition to proof course was designed to ease the transition from calculus-based to upper-level math courses that involved writing proofs. This course was a prerequisite for Linear Algebra, so a variety of STEM (science, technology, mathematics, and engineering) majors were enrolled in this course as well. The first half of the course focused on logic, including direct proof, proof by contradiction, proof by contrapositive, and proof by cases. The second half introduced basic concepts in real analysis, linear algebra, and number theory.

### Data Collection

The four interviews were spread across a semester. Each interview was also task-based, consisting of two proof construction tasks. Participants worked for no more than 15-20 minutes on each proof construction and debriefed their thought process after with the interviewer. All eight tasks were from one content area, basic number theory. Tasks were selected by the researcher to not be heavily dependent on content knowledge nor a singular specific proof technique. Interviews were audio- and video-recorded, and interview notes and student work were collected.

In order to capture their strategies and reasons for using certain strategies, I used a think-aloud protocol (Ericsson & Simon, 1980; Schoenfeld, 1985a), where participants voice their thoughts aloud about a task. Based on the affordances and constraints of asking probing questions (Schoenfeld, 1985a), I minimized interviewer intervention during task performance. Because the phenomenon of interest was the proving *process* itself, keeping the process intact without interruption as much as possible was of the utmost importance.

## Analysis

Qualitative analysis was conducted on video data of participants working on the proof construction tasks. First, video was analyzed for moments when students became stuck. Then, in those moments, I recorded students' strategies, termed proof-specific intentions (Satyam, 2018). Students' strategies were refined using open coding and constant comparison. Lastly, I looked for change in each student's strategies over the eight tasks spanning the semester.

**Indicators of an impasse.** Through watching videos of students' attempts to prove, certain observable behaviors contributed to my judgment of when a student was stuck. A list of these include: silence, no writing, staring at paper, holding paper closer to one's face, sitting back from the paper to look at it as if from a distance, tapping/playing with their pen/pencil, and touching face with hand or pencil. These behaviors were not exhaustive and individuals exhibited different behaviors specific to themselves, but they cover much of what we see when a person is stuck.

## Results: Shifts in How Students Chose A Proof Technique to Use

A common development that occurred across participants was change in how they chose what proof technique to pursue, when trying to construct a proof. By *proof technique*, I mean tools such as direct proof, cases, etc. Proof by contradiction may be referred to here as just contradiction and proof by contrapositive as contrapositive for brevity sake. Eight of eleven participants (pseudonyms used) showed this development, based on interview notes and across all tasks (see Table 1). I discuss two participants here, to illustrate this development.

**Table 1: Select Developments in Proving by Participant**

| | Rationale for a proof technique | Harness awareness of solution attempt | Check examples w/ other strategies | Explore and monitor |
|---|---|---|---|---|
| Amy | | | | X |
| Charlie | X | | X | |
| Dustin | X | | | |
| Granger | | X | X | X |
| Gabriella | X | X | | |
| Joel | | X | | |
| Jordan | X | | | |
| Leonhard | X | | | |
| Stephanie | X | | | |
| Shelby | X | | | |
| Timothy | X | X | X | X |

## Case: Favoring One Proof Technique

Stephanie was chosen here to illustrate the early stages, of where a student uses one proof technique predominantly. From the beginning, Stephanie favored proof by contradiction over all other techniques when constructing a proof. In Interview 1 – Task 1, she jumped to trying proof by contradiction: She immediately identified the assumption as "A," the conclusion of the statement as "B," and wrote the negation.

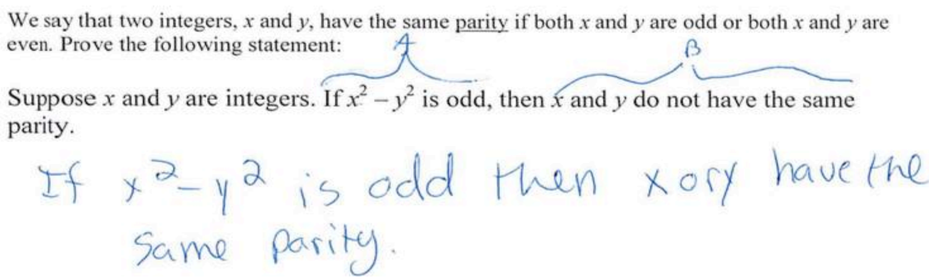


**Figure 2: Beginning of Stephanie's work on Interview 1 – Task 1**

She explained that she used contradiction because the statement was an implication, having an "if-then" structure: "When I see the if-then statement, I immediately think I can do this by contradiction." She explained further that she felt comfortable using this technique. Note that Stephanie technically wrote the negation incorrectly; the correct negation is "A and not B," i.e. "$x^2 - y^2$ is odd and x or y have the same parity." Instead, she wrote the negation as an implication, a common error. However, this error did not affect the rest of her proof and her reasoning for picking contradiction was unaffected by her execution.

In the next interview, Stephanie's go-to method was still contradiction. Upon starting the second task of Interview 2, she said, "I can see that this is an if-then statement, so automatically I'm going to try to use contradiction, but I don't know if it will work or not." She explained that "When I read an if-then statement, I'm most comfortable using negation or a contradiction. So then I just try that, even though I know it doesn't always work, but I just try it." The use of contradiction was automatic for her, saying outright she does not always know if proof by contradiction will lead to a correct solution. The general structure – of a statement having "if" and "then" clauses – was enough to determine that she could use her favored technique, but she did not make use of the statement in any further way to guide her choice of technique.

Stephanie did get stuck on her proof by contradiction, so she switched to contrapositive. She explained during the debrief, "I'll try contrapositive and then I felt a little better after I tried contrapositive just because I thought [out of] both of them, probably one of them was gonna be right." Stephanie did not give a rationale for why, just that it was another technique.

**Summary.** Stephanie's articulations and work during Interviews 1 and 2 show how a student can favor one proof technique and use it whenever they can. Stephanie did have a condition for when to use proof by contradiction, whenever she saw an if-then statement, but this applied to nearly all statements to be proven in the course. Stephanie becomes less dependent on proof by contradiction and her rationale did become more sophisticated over time, but her work was unfortunately incorrect on all four tasks on Interviews 3 and 4 so they are not presented here.

### Case: Recognizing When Best to Use A Certain Proof Technique

We turn now to a different student, Timothy, to see how rationales shift over time. Timothy was similar to Stephanie in having favored proof techniques in the beginning, but his rationales became more sophisticated and based on the statement itself as his interviews progressed, in addition to producing correct or partially correct proofs.

Timothy began his interviews similar to Stephanie in terms of his rationales. Figure 3 shows Timothy's attempt in Interview 1 - Task 1 (same as Stephanie's task). When stuck in the beginning, he re-read the question and wrote what was known. At this point he switched from his direct proof attempt to proof by contrapositive.

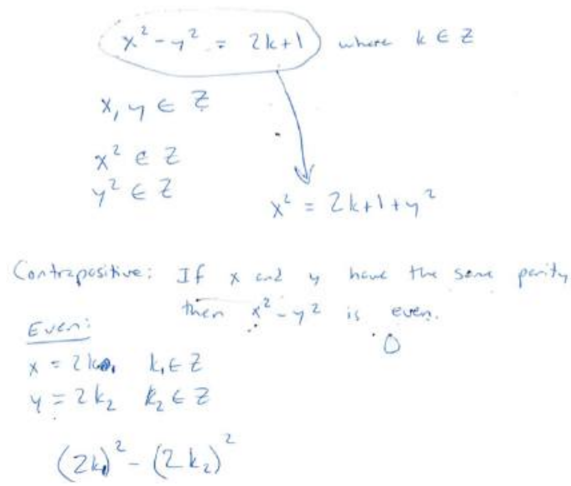

**Figure 3: Beginning of Timothy's work on Interview 1 - Task 1**

When asked why he selected contrapositive, he explained it was a method from class but also that it was a tool logically equivalent to direct proof that he could use:

> Timothy: It was confusing me when I'd try to think of it the normal way so I knew the contrapositive is true, it's basically the equivalent, logical equivalent.
>
> …
>
> Interviewer: So actually, so how did you come up with contrapositive?
>
> Timothy: Looking at it straightforward didn't…it wasn't working for me so I know we learned in class that the contrapositive is basically not B implies not A. I knew we said that was logically equivalent, so if I could prove the contrapositive was true, then I could prove the original statement was true was kinda my thinking with that.

He explained that direct proof was not helpful for generating a proof, but he gave no specific rationale for choosing contrapositive over other proof techniques. His explanation implied that contrapositive was a legitimate tool from class, so why not use it? While it is possible he may have had some internal reason for using contrapositive, he neither mentioned this on his own nor articulated any further reasons when questioned.

Later in this interview, he talked more about contradiction being one of his "go-to" methods and why:

> Timothy: I always go about it with either contradiction or induction or straight up [direct proof] so I kinda knew that I might be able to contradict this never equaling that, so I wrote out the contradiction...I guess contradiction is a little easier for me to think about. You just say the first part of the implication is true and the second part is false. So it's just easier in my head, I guess, just to think about rather than switching around the implication, negating both parts.
>
> Interviewer: Okay.
>
> Timothy: So I guess that's why I go to that first.

Timothy expressed here that contradiction was easier for him than contrapositive, which involves switching and negating both the assumption and conclusion. He did have some rationale for why he might use contradiction, but it was couched in terms of ease of use, first and foremost.

The notion of "ease of use" as determining choice of proof techniques showed up in latter interviews. In his work for Interview 2 - Task 1 (Figure 4), Timothy started by defining x and y using the definition of consecutive numbers but in calculating xy, he became stuck over what to do next. He then switched to contrapositive because "sometimes that's an easier way for me to look at it." He knew that contrapositive was easier on some level for him but not for any reasons specific to the statement and did not further articulate why. Ultimately, his contrapositive proof was not to his liking and also not correct.

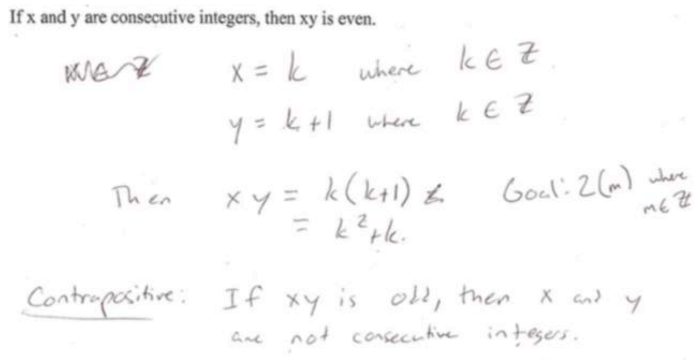


**Figure 4: Timothy's work for Interview 2 – Task 1**

By the end of the interviews, however, Timothy showed sophisticated rationale when considering which proof techniques to use. In Interview 4 - Task 1 (Figure 5), Timothy became stuck after computing the goal, a+b, directly.

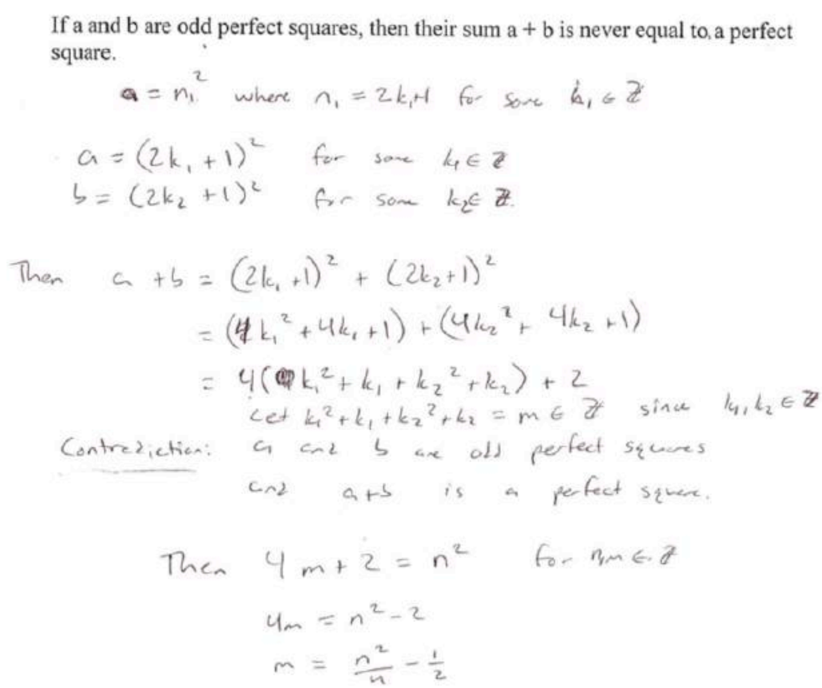


**Figure 5: Timothy's work on Interview 4 – Task 1**

He explained that he used contradiction because "it's easier when I know something like is equal to something or is something." He then gave this further rationale for why contradiction:

> I was trying to prove that it's not equal to a perfect square and I know from past experiences, *it's easier when I know something is equal to something or is something.* So I tried to use contradiction because *I knew I could say then it is a perfect square.*

His argument was that he wanted to be able to work with an equality. Timothy also gave a reason for why he did not use another method, contrapositive:

> I thought about contrapositive, too, but then it would say that A and B are not perfect squares and that's again, like something's *not* so I mean, it's easier for me to work when I know like a straight definition of something. So if I could keep this, I knew if I could keep this, like they are perfect squares and say this is a perfect square, then it'd be easier to work with.

This explanation was similar to his prior one about equality of objects being easier, i.e. knowing things are not equal is not as helpful. His sub goal then was to find a proof technique that would give him a+b *is* a perfect square. This task is also notable for drawing out Timothy's observations on contradiction:

> I never really thought about it this way but I realized when you use the contradiction, you don't really have the assumption and conclusion anymore…*you can actually pick any part of that statement you want and work with it. Rather than with an if/then statement, you start with the assumption and try to work to the conclusion.* So you're not as limited, I guess.

Timothy gave a high-level explanation of the nature of proof by contradiction. He found proof by contradiction freer than other techniques, due to being able to work with all parts of the statement. This is in contrast to starting with the assumption and trying to prove the conclusion, as is done in direct proof but also to an extent proof by contrapositive. Note that this revelation came during this interview context, based on his "I never really thought about it this way but..." clause. The interview served as a vehicle for reflection on proof techniques for Timothy.

**Summary.** Timothy went from picking a proof technique because (1) it existed as a tool, to (2) having a general sense that certain ones would be easier, to (3) explaining how the content of the statement can drive the approach, to (4) articulating understanding at the meta-level of how a technique functions as logical tools. His later interviews revealed insight on when to use contradiction that did not depend on statement content but instead meta-level structure.

## Discussion

Both Stephanie and Timothy showed similar growth in how they chose a proof technique to pursue through most of their interviews. Both discussed liking and being drawn to certain techniques, as their go-to method. Timothy's latter interviews especially illustrated weighing the utility of different techniques, to think about which would be *better*, whether it be a cleaner proof or just easier. He noticed that being able to set things equal provided more to work with and often preferred proof by contradiction for this reason.

The difference between these two cases lies in where they ended: by the end of the interviews, Timothy articulated a general insight for when contradiction was useful. Across all the students, a general trajectory for how students grew in how they chose which techniques to use emerged. Conceptualizing this specific development as a series of stages, Figure 6 illustrates the stages students tended to step through.

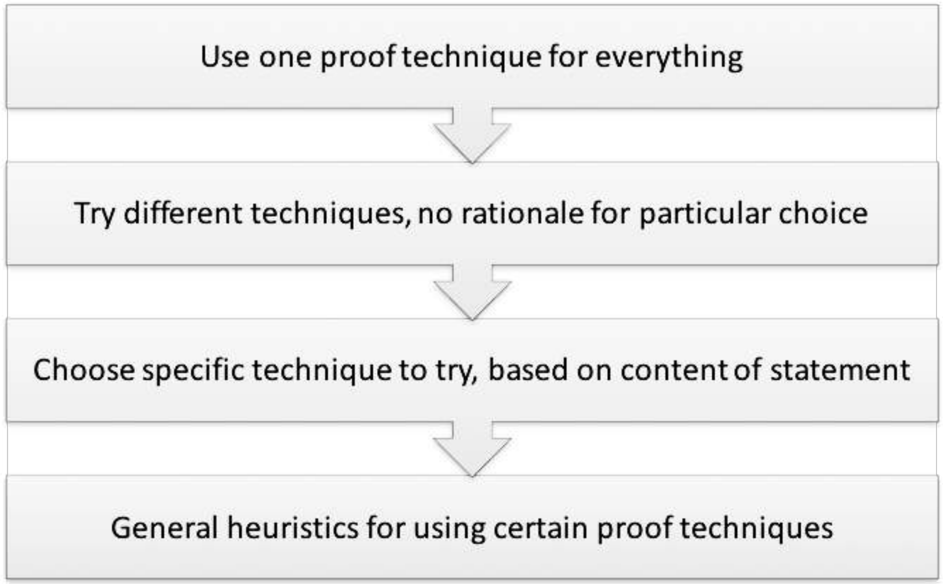


**Figure 6: Stages of development in how students choose proof techniques to pursue**

This development reveals the amount of decision-making that can go into writing even the first line of a proof. Timothy took the content of the statement to be proven into account when deciding how to begin a proof. This suggests a revisiting of the distinction between formal-rhetorical and problem-centered aspects of proving (Selden & Selden, 2007). Acts that we expect to be formal-rhetorical, such as writing the first line of a proof, may retain some of the problem-solving aspects too for students new to proving, as they consider the content as well. It is important that the interplay between these two aspects – formal-rhetorical and problem-centered parts of proving – not be lost when teaching students.

This development is significant because it shows that students do over time grow in their sense of when certain proof techniques are best suited for a problem and that there are general stages. One limitation is that becoming better at using tools is not necessarily reflective of deeper mathematical understanding, as Guin and Trouche (1999) noted about students using calculators as tools. But the cases here shows it is natural for even this kind of judgment to take a while to develop; noticing what proof technique works best for a given statement does not happen instantly but also it must be nurtured. This means that as instructors, we cannot expect students to have this reasoning immediately. Development is of course informed by instruction, so this may be an area that can be supported via instruction, by designing tasks that probe students to consider the strengths and weaknesses of each proof technique. Through this, we can better help students understand and learn how to prove, as a mathematical activity that makes sense.

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