ASSESSING STUDENTS' UNDERSTANDING OF FRACTION MULTIPLICATION

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The purpose of this study is to investigate the range of strategies fifth graders used to solve a word problem involving fraction multiplication. We report a detailed qualitative analysis of elementary students' written work (N = 1472). The results demonstrate that students collectively use a wide range of strategies for fraction multiplication. Implications for teaching and learning are discussed.

Keywords: Number Concepts and Operations; Rational Numbers; Assessment and Evaluation

Perspectives

Kieren (1976) stated five subconstructs to define fractions: part-whole, ratio, quotient, operator, and measure (see also, Behr et al., 1992). A comprehensive understanding of the rational numbers demands students to be familiar with interpretations of various subconstructs as well as understand their interaction (Ball, 1993; Behr et al., 1983; Lamon, 2007, 2012; Ni, 2001). Teaching the algorithm for multiplying fractions seems easy (Johanning, 2019; Reys et al., 2007) but the conceptual underpinnings are complex (Tirosh, 2000; Tsankova & Pjanic, 2009). Usually, additive operations require dealing with one fractional unit, while multiplicative operations involve interaction between multiple units. A problem like 'Ben has 1/3 of a cup of sugar. He sprinkles 1/2 of the sugar onto brownies. How much sugar does Ben sprinkle?' requires – (i) coordination in the units involving '1/3 cup a sugar' with '1/2 of the 1/3 cup' and (ii) choice of a specific arithmetic operation.

This study explores students' conceptions on fraction multiplication for a contextual problem. The results might guide elementary teachers to design strategic problems to capture the implicit conceptions of their students' reasoning. This paper describes the patterns in students' responses to capture the reasons for selecting a specific option by examining their written work. The main question guiding our research is *What is the range and distribution of strategies that students use to approach and solve a fraction multiplication problem?*

Context

We designed a task to address the Grade 5 standard (CCSS.MATH.CONTENT.5.NF.B.4) focusing on students understanding of multiplication related to multiplying a fraction by a fraction (Figure 1). The question included two fractions with different meanings: the fraction 5/8 representing a certain length and functioning *as a measure* of the distance between Levi's home and his school and the fraction 2/3, functioning *as a ratio* between the distance he has walked and the school-home distance. The distractors for this question were purposefully designed to assess the participants choice of operation: (a) 1/24 is the result of subtracting; (b) 10/24 can be obtained by multiplying and is the correct response; (c) 16/24 can be obtained after finding equivalent fractions with a common denominator; and (d) 31/24 can be obtained by adding.

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7. Levi lives $\frac{5}{8}$ mile from school. After he walked $\frac{2}{3}$ of the way to school, he met Marta. How far, in miles, had Levi walked when he met Marta?

A. $\frac{1}{24}$ miles B. $\frac{10}{24}$ miles C. $\frac{16}{24}$ miles D. $\frac{31}{24}$ miles

Figure 1: The Problem

Data and Methods

The data is drawn from a larger study from a representative sample of fifth-grade students (N = 1427) in a Midwestern State. Participation was voluntary, and students were given 15 minutes to work on eight multiple-choice questions. For this paper, we have focused on one question involving fraction multiplication (Figure 1). The written work of the students was examined using qualitative software, MAXQDA version 18.1.1 (VERBI Software, 2016).

Qualitative Analysis of Students' Written Work

Research suggests that students' written work provides valuable evidence of their mathematical strategies, reasoning, and confusions (Brizuela, 2005; Kamii et al., 2001). We used thematic analysis (Braun & Clarke, 2006) to code students' written work. To ensure consistency, two coders coded one class (n = 57) during a training session to identify similarities in students' work and developed a list of themes. Certain pragmatic agreements were made, for example, the code of 'no written work' was used both for blank entries and if the student scratched out or erased their written work. As another example, the code 'unclear explanation' was used if something were written but a clear idea could not be deciphered. The coders used the initial codebook to code three classes individually and agreed on 100% of the cases after discussion. The final codebook had five themes each with several sub-themes (Table 1). Each code was defined in a code book along with prototypical examples to create consistent use.

Results

The unit of analysis for this part of the study is a *student's response to one specific item on fraction multiplication*. To address the research question, we first present the distribution of themes and then summarize the information captured from these themes.

Table 1: List of Themes with Distribution of The Students

	=1472	%
Code 10 Students who selected option 10/24 (**correct response**)	**435**	**9.55**
• 10A: Used multiplication as operator ('*' or '.' or 'of')	157	0.67
• 10A(a): Reduced the fraction to 5/12	36	2.45
• 10A(b): Used multiplication as second operator choice	7	0.48
• 10B: Used drawings	2	0.14
• 10C: Used "wrong" or "no" arithmetic operator	13	0.88
• 10D: Incorrectly written work	2	0.14
• 10E: No written work	145	9.85
• 10F: Unclear explanation	41	2.79
• 10F(a): Showed understanding of making the same denominators	14	0.95
• 10F(b): Subtract fractions and select the one with same multiples	6	0.41
• 10F(c): Added fractions as 7/11 or 7/24	11	0.75
• Guess	1	0.07

Code 1 Students who selected option 1/24	**414**	28.13
• 1A: Made the denominators same and subtracted both	125	8.49
• 1A(a): Subtracted fractions but did not write '-' in their work	36	2.45
• 1A*: Made the denominators same, flipped the numbers and subtracted	119	8.08
• 1B: Unclear explanation	54	3.67
• 1B(a): Subtracted fractions and have written 3/5 as answer	6	0.41
• 1B(b): Used other/multiple operations	6	0.41
• 1C: No written work	68	4.62
Code 16 Students who selected option 16/24	**247**	6.78
• 16A: Made denominators same; selected the one with large numerator	60	4.08
• 16A(a): Compared fractions but used '-' in their written work	10	0.68
• 16B: Unclear explanation	36	2.45
• 16B(a): Used multiple/other operators	11	0.75
• 16B(b): Transformed either 5/8 to 15/24 or 2/3 to 16/24	17	1.15
• 16C: No written work	113	7.68
Code 31 Students who selected option 31/24	**158**	0.73
• 31A: Made the denominators same and added both	97	6.59
• 31A(a): Added fractions but did not write '+' OR wrote '-'	22	1.49
• 31B: Unclear explanation	9	0.61
• 31B(a): Added numbers as 7/11	2	0.14
• 31C: No written work	28	1.90
Code O Other Responses	**218**	4.81
• O(A): Students have written 'what' or 'IDK' or 'guessed' or '?'	2	0.14
• O(B): No response with unclear or clear written work	23	1.56
• O(C): No response and NO written work	193	3.11

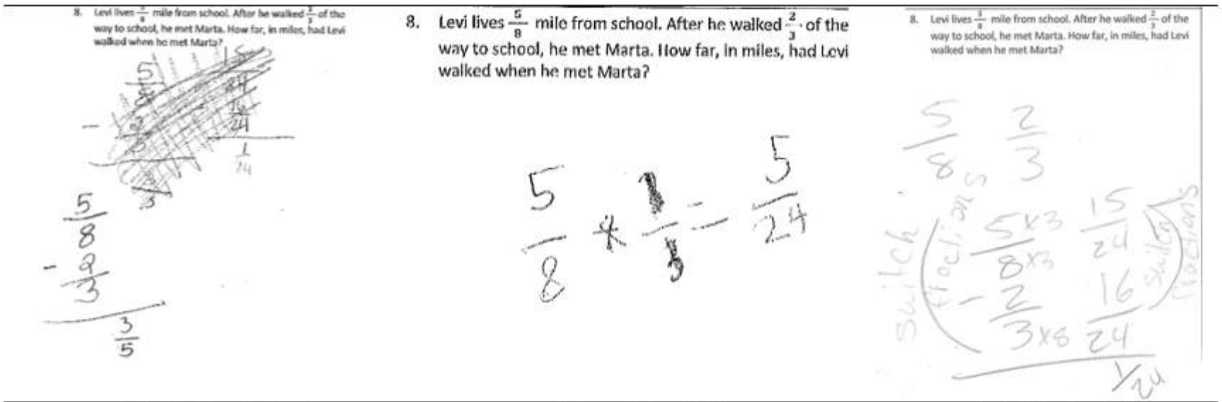
N: Number of Students; %: Percentage of Students

The students' written work revealed their comprehension and conception of a fraction multiplication word problem. Around 435 students (29.55%) chose the correct option 10/24, but 62 (4.21%) of them had unclear explanations. Some students did not use the multiplicative operator as their first choice and employed a 'guess and check' strategy solving the task, e.g., making the denominator values the same, adding the fractions, etc. as their first attempt (**Code 10A(b)**, n = 7, 0.48%). **Code 10C** depicts the students (n = 13) who have either not used any or used '-' as an arithmetic operator between 5/8 and 2/3. The reason of their selecting their operation is unknown to us but suggests avenues to be explored in the future using interviews.

Some students (**Code 10F(b),** n = 6, 0.41%) wrongly subtracted the fractions 5/8 and 2/3 as 3/5, and selected 10/24 (Figure 2(a)). A potential reason for this selection can be that the students might have realized that 3 and 5 are respective factors of 24 and 10. There is speculation in this inference but the best judgment we can make from their work. However, this code supported the idea of revisiting previously learned concepts because even if the subtracting fractions is a fourth-grade standard (CCSS.MATH.CONTENT.4.NF.B.3.A), misconceptions were visible in their present work. Similar reasoning has also been captured in **Code 1B(a) (**n = 6, 0.41%).

Some students demonstrated an advanced level of understanding by treating fractions as an operator. They considered 2/3 as 2*(1/3), multiplied 5/8 with 1/3 to get 5/24, and then doubled the output (Figure 2(b); **Code 10A**, n = 157, 10.67%). This shows a sophisticated level of reasoning as students changed the fraction into a unit fraction and then doubled it.

Many students chose the option 1/24 (**Code1;** *n* = 414, 28.13%) suggesting that they relied on a part-whole understanding of fractions. Students subtracted 5/8 and 2/3 after making the same denominator; this only makes sense if they were treating both as a distance. **Code1A*** (*n* = 119, 8.08%), an extension to **Code1A** (related to subtracting both fractions, *n* = 161, 10.94%), reflected a unique attribute of understanding where students made the denominators the same for both fractions and switched their order (Figure 2(c)). This code indicated students may have applied the rule of always subtracting the smaller number from the larger number. Similar reasoning was captured by **Code 31** (*n* = 158, 10.73%) where students interpreted this to be an addition problem.



(a) Code 10C(d) (b) Code 10A (c) Code 1A*

Figure 2: Examples of the Students' Work for Specific Codes

A few codes call for teachers' attention, for instance, **Code 16B(b)** (*n* = 17, 1.15%) depicts students using only one fraction to deduce the answer. One of the students mentioned, "I think it is 16/24 because 1/3 = 8/24 and he is 2/3, so you add and get 16/24". We also found that there were students who performed the correct calculation and obtained 5/12 but did not select any option, captured in the **Code O(B)**. Such students might have forgotten to mark an answer choice or potentially did not recognize equivalence between 5/12 and 10/24.

Discussion

The qualitative analysis of the students' written work helped in recognizing and identifying the strategies involved in answer choices. We find it plausible that students selected incorrect options because to answer correctly requires one to understand the different meanings of fractions and how to interpret the product (Wyberg et al., 2012). Instead, students often rely on their understanding of whole numbers (Tsankova & Pjanic, 2009; Wu, 2001). However, whole number strategies are not appropriate for finding the product of two fractions. The analysis shows that many students added or subtracted directly which implies they treated both fractions in the question (5/8 of a mile and 2/3 of the distance) as a measure. The students in this sample may lack understanding of a fraction as a ratio.

Previous researchers mentioned that instruction in the elementary classrooms is dominated by the part-whole interpretation of fractions (Ni & Zhou, 2005; Olanoff et al., 2014). Such instruction does not provide the conceptual understanding necessary to solve a problem involving fractions with a ratio meaning. We speculate that reinforcing the conceptual meaning behind all fraction sub-constructs might improve students' facility with fraction operations.

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