INVESTIGATING THE LEARNING SEQUENCE OF DECIMAL MAGNITUDE AND DECIMAL OPERATIONS

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Research is mixed on whether understanding decimal magnitude supports operations with decimals or whether operations can be learned before and while students develop understanding of decimal magnitude. In the present study, we used a large scale, longitudinal design to investigate students' knowledge of decimal comparison and operation before and after decimal comparison alone was introduced in the curriculum. Student performance on a decimal comparison task did not increase, but there was an increase in performance on decimal subtraction and decimal multiplication tasks, topics which were not part of the mandated curriculum during the relevant period of instruction.

Keywords: Number Concepts and Operations, Rational Numbers, Elementary School Education, Cognition

Perspectives

Researchers examined various conceptual hurdles involved in meaningful interpretation and use of the notational system involving decimals (Resnick et al., 1989). Hiebert (1992) proposed three types of knowledge which are important to comprehend the decimal system: knowledge of the notation, knowledge of the symbol rules and knowledge of quantities and actions on quantities. The knowledge of notation comprises of "how the symbols are positioned on paper" (ibid, p. 290) rather than understanding of what '.' means or what quantities it represents. For instance, a student can compare two decimals correctly, but can have incorrect reasoning to explain their answers (see Resnick et al., 1989 for details on erroneous rules while comparing decimal numbers). The knowledge of the symbol rules prescribes on "how to manipulate the written symbols to produce correct answers" (Hiebert, 1992, p. 290). For instance, while adding and subtracting two or more decimals, the numbers need to be lined up systematically (Lai & Murray, 2015). This knowledge is analogous to Skemp's (1976) idea of instrumental understanding where an individual can manipulate mathematical syntactic symbols using appropriate rules, procedures, algorithms, etc. to produce the correct answer, even when without understanding the underlying reasons. Knowledge of quantities and actions includes the understanding of decimal numbers are representing quantities, i.e., measures of objects "…by units, tenths of units, hundredths of units, and so on" and comprehending the reasons that explain "what happens when the quantities are moved, partitioned, combined, or acted upon in other ways" (Hiebert, 1992, p. 291). Lai and Murray (2015) related the knowledge of quantities and actions on quantities with developing a comprehensive understanding of the decimal topics.

Decimal Comparison

Students build on whole number ideas when they engage with decimals, and this both helps and hinders learning. Lee and colleagues (2016) argued that due to the representational nature of decimal numbers, which is virtually indistinguishable from that of whole numbers, the students find decimal magnitude comparison tasks easier as compared to the fraction magnitude (see also, DeWolf et al., 2014; Iuculano & Butterworth, 2011). Researchers claim that students often perform well on the decimals comparison tasks by following syntactical rules (Lachance & Confrey, 2002), rather than developing a conceptual understanding of it. However, the common practice of teaching decimals as

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an extension of the whole numbers might convey an inadequate understanding of place-value system (Fuson, 1990; Martinie, 2014).

Sequence of Decimal Instruction

In the United States, decimal instruction begins in the fourth grade with comparisons of fractions written in decimal form with denominators of 10 and 100. A decimal is regarded as the one-dimensional magnitude of a fraction (a/b = c) expressed in the form of the standard base-10 metric system (Lee et al., 2016). This continues in fifth grade with decimal operations to the hundredths place (National Governors Association Center for Best Practices, 2010). Decimal instruction then continues through middle school (Rittle-Johnson et al., 2001).

Although curricula frequently sequence decimal comparison instruction before operations, there is little research available to support this sequence. The magnitude-before-computation sequence is supported for fractions and standards and curricula appear to follow it for decimals based on the idea that fractions and decimals are closely conceptually related, even though there has been very little research on this instructional order. Arguments for teaching decimal magnitude before tackling operations between decimals numbers are formed by research that shows that children who are less comfortable with fraction magnitudes are also not as good as their counterparts at computations involving fractions (Lortie-Forgues et al., 2015). Other researchers note that students can understand decimal magnitude without being able to understand the results of computations involving decimals, which implies that understanding decimal magnitude is a prerequisite for decimal operations (Siegler & Lortie-Forgues, 2015).

Decimal Comparison and Operations with Decimals

Even though decimal magnitudes are taught first and operations second, the concepts appear to be intertwined in the minds of students. Decimals are familiar to students before they reach fourth grade to some degree because they follow some of the whole number rules, even though students often misapply those rules for comparing and computing with decimals (Ren & Gunderson, 2019; Rittle-Johnson et al., 2001; Vamvakoussi & Vosniadou, 2004). As students make sense of decimals in school, they begin to apply what they understand to computation even if they have not been explicitly taught to do so. Hiebert et al. (1991) showed that children could learn about decimal concepts and structure and still show growth on decimal computation with symbols without detailed instruction on procedures. Other research has found that intermingling work on decimal place value with decimal addition and subtraction results in strong student performance, which is opposite of the assumptions surrounding magnitude-first instruction (Rittle‐Johnson & Koedinger, 2009). Given the mixed findings from past research that has mostly relied on small-scale qualitative data, in the present study, we used a large scale, longitudinal design to investigate students' knowledge of decimal comparison and operation before and after it was introduced in the curriculum. In particular, we sought to answer two research questions. (1) *How does knowledge of decimal comparison and operations with decimal change during the year in which decimals are formally introduced in the curriculum?* (2) *Are the patterns that characterize students' responses at each time point more indicative of magnitude-before-operation or intermingled learning in the decimal domain?*

Methods

The data is drawn from a larger study that included a representative sample of Grade 4 elementary teachers in Indiana. These teachers administered 8-item tests to their Grade 4 students (N = 1467) in the Fall of 2017 and Spring of 2018, and the data we report comes from three items on that test. The participation in this survey was voluntary for the students and they were given 15 minutes to work on the test. McNemar's test was used to compare the pre-test and post-test results of the same students in grade four at two different points in the school year, so we had matched pairs of subjects with a dichotomous trait of correct or incorrect for each question. We used alluvial diagrams to search for patterns in pretest and posttest responses.

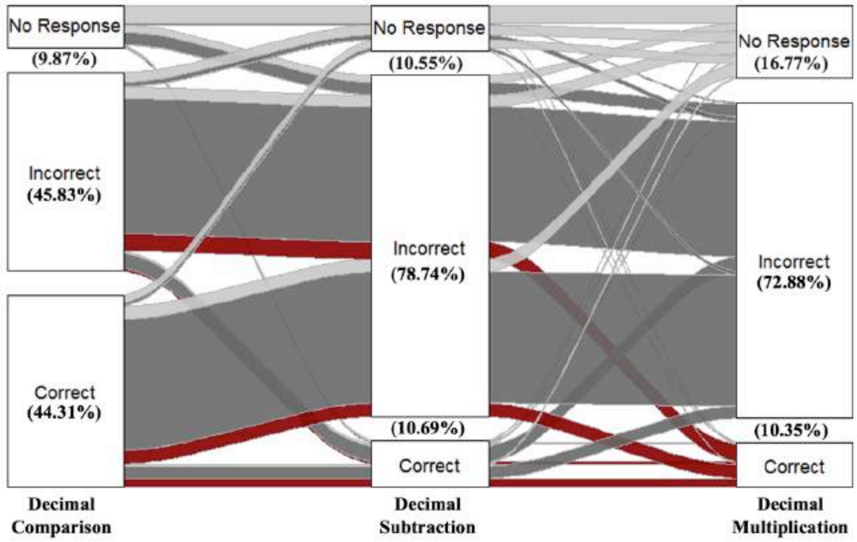

Figure 1. Pretest weighted flow of participants between response categories.

Results

An exact McNemar's test was used to compare the two conditions (correct and incorrect) on the pre- and post-test over fourth-grade fraction and decimal knowledge. The change in the number of students who correctly answered the comparison question from the pretest to the posttest was small, 44.31% to 45.07%. The analysis showed that there was not a statistically significant positive change between the pre- and post-test for ordering decimals from smallest to largest (p = 0.689). Of the 1,467 students who took the pre- and post-tests, 642 answered the ordering question correctly on the pre-test and 653 answered correctly on the post-test.

In contrast, there was a larger change in the number of students answering the decimal subtraction question correctly on the posttest from 10.67% to 28.57%, and this change was a statistically significant (p = 0.000). For decimal addition, 155 students answered correctly on the pre-test and 414 answered correctly on the post-test. For the decimal multiplication question, the increase was more modest, from 10.35% answering correctly on the pretest to 17.53% answering correctly on the posttest. Similar to the decimal subtraction problem, this increase statistically significant (p = 0.000). In the decimal multiplication question, 150 students answered correctly on the pre-test and 254 answered correctly on the post-test. The students' performance differences between the decimal magnitude question and the operation questions shows that as a group the students improved in their ability to operate on decimals without substantially increasing their understanding of decimal comparison.

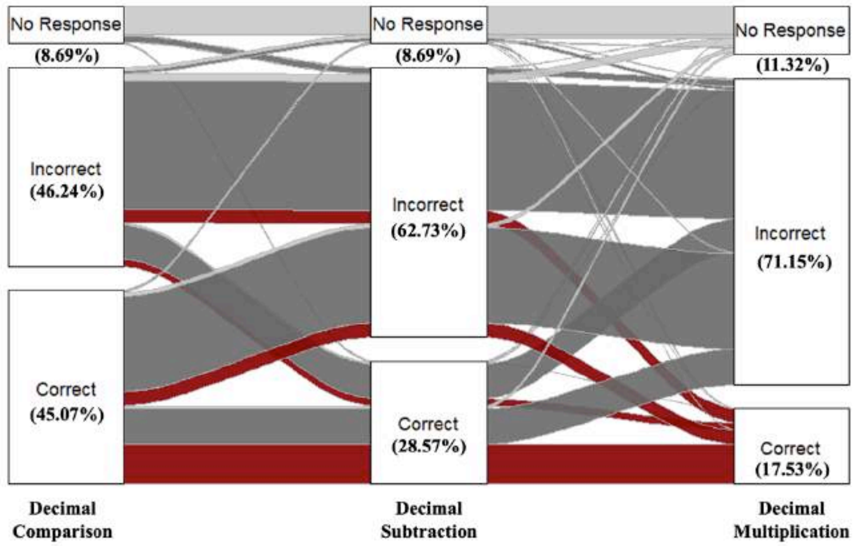

Figure 2. Posttest weighted flow of participants between response categories.

We examined the relationship between responses on these three items and compared these patterns of responses between the pretest and posttest. The alluvial diagrams below show weighted flows among the three decimal questions on the pre- and posttests from the first decimal question about comparing decimals, through the subtraction question, and then to the multiplication question. These diagrams illustrate how responses to decimal items early in the test were related to responses later in the test at each time point. In particular, although the number of students who answered the comparison question did not significantly increase at posttest, a much larger portion of those students went on to answer the two operation questions at posttest than at pretest (see large ribbon at the bottom of Figure 2).

Discussion and Implications

We expected that fourth grade students would show more growth on decimal comparisons than decimal operations in fourth grade because decimal comparison is a fourth-grade standard and decimal operations are a fifth-grade standard. Students were presumably receiving more instruction on magnitude comparisons than on decimal operations. What we saw instead was that growth in decimal comparisons was not statistically significant yet growth in decimal operations (subtraction and multiplication) were statistically significant. Furthermore, by comparing the patterns between items at each time point we noticed that a majority of the students who answered the multiplication question at posttest also answered the comparison and subtraction problems, suggesting most of the change from pre to post was driven by a cohort of students who solidified their understanding of operations during the year in which comparison was taught. These findings confirm at scale what other researchers have found in small, qualitative studies; namely, that the conceptual development of both comparisons of magnitude and operations happen concurrently rather than sequentially.

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