"H IS NOT A NUMBER!" EXAMINING HOW NUMBER INFLUENCES VARIABLE

Karen Zwanch
Oklahoma State University
karen.zwanch@okstate.edu

Students' conceptions of variable do not always support normative interpretations of equations, and research links limited conceptions of variable to operations on composite units (i.e., units of units). This study examines how one ninth grade Algebra 1 student, Alex's, concept of number is related to his conceptions of variables when writing and interpreting linear equations and expressions. Alex had constructed an advanced tacitly nested number sequence (aTNS). An aTNS is the third stage out of five in the number sequence hierarchy, and indicates that he operates on composite units but does not reason multiplicatively. Analysis links the cognitive structures that define Alex's aTNS to his applied conceptions of variable, and finds that non-normative conceptions of variable manifest due to limitations operating on composite units. Being constrained to additive reasoning limited Alex's use of variables in multiplicative situations.

Keywords: Algebra and Algebraic Thinking; Number Concepts and Operations

Understanding how symbols are used in mathematics is critical to students' success, but algebra curriculum tends to focus more on manipulating symbols than it does on the meaning of symbols (Sherman, Walkington, & Howell, 2016). Letters are particularly difficult for students (Bush & Karp, 2013) because they can represent one number (e.g., e); one unknown quantity (as when solving for x); a "pattern generalizer" (e.g., representing odd numbers as $2n + 1$; Usiskin, 1988, p. 9); or a varying quantity (e.g., $A = l \cdot w$; Baroody, 1998). Students must discern these four uses of letters in algebra in order to make sense of algebraic expressions and equations.

With these four meanings in mind, it is unsurprising that variable misconceptions have been well documented. Knuth, Alibali, McNeil, Weinberg, and Stephens (2005) categorized the ways in which students in grades six through eight conceived of variables. The categories included conceiving of variables as representing multiple values, a specific value, or an object. In this study, the percentage of students who responded that n represents multiple values in an expression like $3n$ increased from less than 50% in sixth grade to more than 75% in eighth grade.

MacGregor and Stacey (1997) similarly documented higher instances of variable misconceptions in later years of schooling (years 8-10) compared to earlier (year 7). Stacey and MacGregor's (1997) research also generated categories for students' misconceptions, including: abbreviated word, alphabetical value, numerical value, use of different letters, letter ignored, labels, variable equals one, and general referent. They found that interference from new learning and poor instruction may account for increased variable misconceptions in older students.

Hackenberg, Jones, Eker, and Creager (2017) have studied the mental structures that support students' conception of variable. They found that operating on composite units (i.e., units of units) supports students' conception of variable as an unknown because a quantitative unknown consists of a composite unit containing an unknown number of units of one. This directly ties students' conceptions of variable to their operations on composite units. Zwanch (2019, 2020) also found that operations on composite units and multiplicative reasoning are related to students' interpretations of linear equations. The present study examines how an Algebra 1 student's conceptions of variable are related to his concept of number.

In: Sacristán, A.I., Cortés-Zavala, J.C. & Ruiz-Arias, P.M. (Eds.). (2020). *Mathematics Education Across Cultures: Proceedings of the 42nd Meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education*, Mexico. Cinvestav / AMIUTEM / PME-NA. https:/doi.org/10.51272/pmena.42.2020

Theoretical Framework

Students' units construction and coordination (Ulrich, 2015, 2016) specifies that the levels of complexity with which they interpret and operate on number is related to their ability to create different levels of units. These levels of units inform students' number sequences (Steffe & Olive, 2010; Ulrich, 2016b), which are the "recognition template of a numerical counting scheme. … [C]hildren use their number sequences to provide meaning for number words" (Steffe & Olive, 2010, p. 27).

Tacitly Nested Number Sequence (TNS)

The TNS is the second of five in the number sequence hierarchy (Steffe & Olive, 2010). TNS students construct composite units in mental activity. To construct composite units in activity means that TNS students interpret a number word, such as "seven," as seven individual units, or counting acts (Ulrich, 2015). Thus, "seven" is thought of as the seven numbers from one through seven, or from 34 through 40, for instance. TNS students can engage in mental activity to construct a composite unit of seven, but can neither operate nor reflect on the composite unit.

Advanced Tacitly Nested Number Sequence (aTNS)

In the number sequences, an aTNS is the third stage out of five, and it is characterized by assimilatory composite units (Ulrich, 2016b). An *assimilatory composite unit* allows students to interpret a number word, such as "seven," as a single unit containing seven individual units of one (i.e., seven is one unit of seven units), and supports operations on composite units (Ulrich, 2016a). This means that aTNS students can construct a third level of units in mental activity (e.g., 21 as a unit containing three units of seven). Following activity, the third level of units decays leaving aTNS students to reflect on 21 as a composite unit containing 21 units of 1. An assimilatory composite unit advantages aTNS students' numerical reasoning over TNS students, but aTNS students remain limited to additive reasoning.

Explicitly Nested Number Sequence (ENS)

An ENS is the fourth stage in the number sequence framework (Ulrich, 2016b; c.f. Steffe & Olive, 2010). Like aTNS students, ENS students also assimilate with composite units and can construct three levels of units in mental activity. However, ENS students have also constructed multiplicative reasoning that supports them in thinking of seven, for instance, as a unit that is seven times the size of a unit of one.

This research study examines an aTNS student's conception of variable. Specifically, this research asks: (1) What concepts of variable does the student apply when writing and interpreting linear equations and expressions? and (2) In what ways does operating on composite units but not reasoning multiplicatively support or limit his concept of variable?

Methods

This study was conducted in a middle and high school in the rural southeastern United States. 326 students across grades six through nine were given a survey. The purpose of the survey was to attribute a number sequence to each student (Ulrich & Wilkins, 2017). Based on the number sequences attributed by survey analysis, 18 students participated in semi-structured clinical interviews. Each student was interviewed on two days, for approximately 45 minutes each day. The first portion of the interviews confirmed the students' number sequence. Questions were taken from the methods of Ulrich and Wilkins (2017). The next portion of the interviews characterized student's algebraic reasoning. Tasks will be described in the results.

The participant reported here was a ninth grade, Algebra 1 Part 2 student named Alex (a pseudonym). Algebra 1 Parts 1 and 2 was a two-semester course that covered the content of high school algebra. Alex was a tenacious problem solver and was quick to explain his thinking. Alex was identified as an aTNS student by the survey.

Results & Analysis

Alex reasoned about a variable as an object on four of the 11 algebra tasks. One such task asked him to write an equation to represent the following situation: “This week the soccer team scored three fewer points than they did last week.” In this situation, he was told that last week’s and this week’s scores were unknown. Alex wrote $LW - 3$, and this explanation followed:

> Alex: Last week minus three. … Last week minus three equals fewer? I don’t know. Fewer points? … [If] last week they scored 10 and this week they scored less than three, that would be, like, seven.
> Interviewer: … Can we put a variable on that side of the equation?
> Alex: Yeah. $7p$ for points. (Writes $LW - 3 = 7p$)
> I: So does that work, then, if they scored, let’s say five points last week?
> Alex: … Five minus three equals two. Ok, think. It’d be two points this week. (Writes $LW - 3 = 2p$ on the next line.)

Alex was able to conceive of LW, which stood for last week’s score, as an unknown quantity in the context of the expression $LW - 3$. Consistent with Hackenberg et al.’s (2017) conclusion that an unknown comprises a composite unit, Alex’s aTNS supported his conception of LW as a composite unit containing an unknown number of units of one. His aTNS also supported additive operations on LW, evidenced by the expression $LW - 3$.

Alex could not, however, think of $LW - 3$ as an entity equal to a second unknown. He said that it equaled fewer points. To conceive of $LW - 3$ in relation to this week’s score, Alex needed to construct a three-level unit structure. This was supported by Alex’s additive operations on composites, however, following the mental operations, the third level of units decayed leaving Alex to reflect only on $LW - 3$ as representing “fewer points.” Then, Alex initiated a numerical example in which $LW = 10$. Substituting specific numbers for unknowns decreased the complexity of the unit structure, allowing Alex to think about $LW - 3$ as $10 - 3$, which he could equate to seven. When asked to incorporate a second variable, however, he said, “$7p$ for points.” This is evidence of his conception of p as a label on 7, rather than an unknown.

Finally, the interviewer attempted to perturb Alex’s thinking by asking if the equation $LW - 3 = 7p$ would work if the team scored five points last week. Alex was not perturbed, however, and wrote a second equation rather than recognizing the limitation of the equation he had written. This is evidence that conceiving of the additive relationship between two unknowns is beyond the limits of Alex’s algebraic reasoning. His aTNS supported conceiving of one variable as an unknown, and supported additive operations on the unknown; it did not support his normative inclusion of a second unknown into the equation because an aTNS does not support reflection on a third level of units.

At another point during the interview, Alex was asked to represent the weight of a $\frac{1}{24}$ share of a candy bar, given that the candy bar weighed h ounces (adapted from Hackenberg & Lee, 2015). Alex determined two numerical examples (24:1 and 48:2), but he could not think about h as the unknown weight of the candy bar.

> Interviewer: [Can you] represent the weight of just your piece if the whole thing weighs h?
> Alex: h is not a number!
> Interviewer: (Surprised) You said, “h is not a number?”
> Alex: Yeah! That’s not a number!

Later, Alex redirected the discussion back to h. When asked if he understood the task, he said, “Yes and no. The ‘no’ is how you got a number from a letter. … I don’t think you can say that. You can’t say that a candy bar equals h ounces, cause it’s not a number.”

The candy bar task is different because it required Alex to conceive of the multiplicative relationship between two unknowns. In a multiplicative context he could not conceive of h as an unknown because he could not construct and reflect on the three-level unit structure representing the relationship between the whole candy bar and his piece. This is a limitation of his aTNS that manifested differently on the multiplicative task than on the additive tasks. Rather than Alex conceiving of the variable as an object, he insisted that the variable not be present at all, and that the relationship can only be represented numerically. Reasoning about numerical examples instead of unknowns reduced the complexity of the unit structure, thereby allowing Alex to construct and reflect on 24 units of 1, rather than h as 24 units of an unknown number of ones.

Discussion

Alex is a ninth grade aTNS student in the second semester of an algebra class. Regardless, he could not write one-step equations to represent additive or multiplicative relationships. The first research question asked what concepts of variable Alex demonstrated. When writing additive expressions, Alex reasoned about a variable as an unknown, but when asked to reason about the relationship between the expression $LW - 3$ and this week’s score, Alex reverted to numerical examples and interpreted variables as objects.

The inability to establish the relationship between two variables when writing an equation is a persistent difficulty for high school algebra students (Bush & Karp, 2013), and the results of the second research question provide a theoretical lens to understand the complexity that contributes to this difficulty for aTNS students. aTNS students assimilate with composite units and can perform additive mental operations on composite units. Such operations supported Alex’s expression writing. However, the results of the mental operations decay following activity. This manifested behaviorally in Alex’s inability to reflect on the relationship between the expression and the second unknown. To compensate for this limitation, Alex reverted to numerical examples, and interpreted the variables in his equations as labels on numbers.

On the candy bar task, Alex did not introduce a variable at all. Hackenberg and Lee (2015) found that students who assimilate with composite units and reason multiplicatively may represent the unknown weight of the smaller piece of the candy bar as $\frac{h}{24}$. Alex was not able to write $\frac{h}{24}$, and concluded that “h is not a number!” This illustrates the manner by which aTNS students, who assimilate with composite units and are constrained to additive reasoning, may be more limited in comparison to their peers who assimilate with composite units and reason multiplicatively (i.e., ENS students).

On the tasks presented here, Alex reasoned with non-normative concepts of variables or did not include variables. These results bring to light the importance of studying not only students’ operations on composite units as prerequisite cognitive structures to their algebraic reasoning, but also their construction of additive versus multiplicative reasoning. Furthermore, Alex’s solutions demonstrate that he is capable of reasoning algebraically to some extent. Thus, research should continue to examine the ways in which algebra instruction can productively support aTNS students’ concept of variable and their additive and multiplicative operations on variables.

References

Baroody, A. J. (1998). *Fostering children’s mathematical power: An investigative approach to K-8 mathematics instruction.* Mahwah, NJ: Erlbaum.

Bush, S. B. & Karp, K. S. (2013). Prerequisite algebra skills and associated misconceptions of middle grade students: A review. The Journal of Mathematical Behavior, 32, 613–632.
Hackenberg, A. J., Jones, R., Eker, A., & Creager, M. (2017). “Approximate” multiplicative relationships between quantitative unknowns. *The Journal of Mathematical Behavior, 48*, 38–61.
Hackenberg, A. J. & Lee, M. Y. (2015). Relationships between students’ fractional knowledge and equation writing. *Journal for Research in Mathematics Education, 46*(2), 196–243.
Knuth, E. J., Alibali, M. W., McNeil, N. M., Weinberg, A., & Stephens, A. C. (2005). Middle school students’ understanding of core algebraic concepts: Equivalence & variable, *ZDM, 37*(1), 68–76.
Küchemann, D. (1981). Algebra. In K. Hart (Ed.), *children’s understanding of mathematics: 11-16* (pp. 259–276). London: Murray.
MacGregor, M. & Stacey, K. (1997). Students’ understanding of algebraic notation: 11-15. *Educational Studies in Mathematics, 33*(1), 1–19.
Sherman, M. F., Walkington, C., & Howell, E. (2016). A comparison of symbol-precedence view in investigative and conventional textbooks used in algebra courses. Journal for Research in Mathematics Education 47(2), 134–146.
Steffe, L. P. & Olive, J. (2010). *Children’s fractional knowledge.* New York, NY: Springer.
Ulrich, C. (2015). Stages in constructing and coordinating units additively and multiplicatively (part 1). *For the Learning of Mathematics, 35*(3), 2–7.
Ulrich, C. (2016a). Stages in constructing and coordinating units additively and multiplicatively (part 2). *For the Learning of Mathematics, 36*(1), 34–39.
Ulrich, C. (2016b). The tacitly nested number sequence in sixth grade: The case of Adam. *The Journal of Mathematical Behavior, 43*, 1–19.
Ulrich, C. & Wilkins, J. L. M. (2017). Using written work to investigate stages in sixth-grade students’ construction and coordination of units. *International Journal of STEM Education, 23*(4). Retrieved from https://stemeducationjournal.springeropen.com/articles/10.1186/s40594-017-0085-0
Usiskin, Z. (1988). Conceptions of school algebra and uses of variables. In A. F. Coxford (Ed.), *The ideas of algebra, K-12* (pp. 8–19). Reston, VA: National Council of Teachers of Mathematics.
Zwanch, K. (2019). Using number sequences to model middle-grades students’ algebraic representations of multiplicative relationships. In Otten, S., Candela, A. G., de Araujo, Z., Haines, C., & Munter, C. (Eds.) *Proceedings of the 41st Annual Meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education* (pp. 177–185). St. Louis, MO.
Zwanch, K. (2020). Using number to support students’ generalizations. In Cribbs, J., & Marchionda, H. (Eds.) *Proceedings of the 47th Annual Conference of the Research Council on Mathematics Learning* (pp. 34–41). Las Vegas, NV.