PSEUDO-EMPIRICAL, INTERNALIZED, AND INTERIORIZED COVARIATIONAL REASONING

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In this brief theoretical report, I describe the process of the construction of units coordinating structures as the result of a non-linear progression from pseudo-empirical to internalized to interiorized mental activity, and I propose the utility of a parallel distinction between pseudo-empirical, internalized, and interiorized levels of covariational reasoning.

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Research with middle grades students suggests that the attributes of a quantity conceptualized by a student can lead to assimilation to schemes involving different units coordinating structures (Boyce & Norton, 2017). Similarly, I expect that when analyzing students' covariational reasoning, a critical aspect of their reasoning is *how* they assimilate attributes of objects as measurable and *how* they assimilate quantities as co-varying. In this paper, I propose utility of adopting distinctions between students' schemes for coordinating units (pseudo-empirical, internalized, and interiorized) to levels of covariational reasoning (Thompson & Carlson, 2017). I begin by providing background on scheme theory (von Glasersfeld, 1995).

Scheme Theory

A scheme consists of three parts: recognition of a situation, operations (mental actions), and an expected result (von Glasersfeld, 1995). Following von Glasersfeld (1995), I distinguish three types of schemes based on their activity: *pseudo-empirical, internalized, and interiorized* schemes. The "empirical" part of a pseudo-empirical scheme refers to an individual's need for an external object of attention to act upon; the activity portion of the scheme requires sensory-motor experience of an external transformation. What makes it "pseudo"-empirical rather than empirical is that the object acted upon is figurative material; the result of the scheme is not about the object itself. With an *internalized* scheme, perception of an act of transformation is still required, but the transformation can involve completely imagined representations (i.e., mental imagery). Representations of the results of internalized schemes can still involve external representations, but actions with these external representations involve communicating internalized reasoning rather than being a necessary aspect of one's reasoning. Both internalized and pseudo-empirical schemes involve mental activities that are experienced temporally; as part of a flow of experience of perceiving an object, acting upon it (mentally), perceiving the resulting object, and conceiving the results of the action. In contrast, an *interiorized* scheme does not require either internal or external representations for mental activity. Interiorized schemes are *anticipatory*, in the sense that the recognition of a situation, the mental actions, and the expected result of the actions of an interiorized scheme are experienced as synchronous, reversible, and necessary.

Although the process of interiorization is prefaced by stages of pseudo-empirical and internalized activity, constructing more advanced schemes is dependent upon individuals' lived experiences rather than following a strictly linear process, via psychological processes of *perturbation*, *abduction*, *assimilation*, *accommodation*, and *reflective abstraction*. Perturbation is the experience of a lack of stability or reliability of one's current schemes; often accompanied with emotive experiences of uncertainty or confusion (Piaget, 1970). Perturbation can be momentary and is most often closely tied to social interactions (communication with others about their mathematical reasoning can be

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viewed as a process of introducing and resolving perturbations involving interpreting others' semiotics; Steffe & Thompson, 2000). Perturbation can also be prolonged and invoke a powerful *intellectual need* for resolution (Harel, 2013) and involve internalized communication (Sfard 2007).

Abduction is a logical process of forming a hypothesis that, if true, would be experienced as satisfying an observation (Norton, 2008; Prawat, 1999). The process of assimilation is the result of a successful abduction of a modification of recognition of a situation or a modification of the recognition of a result that resolves a perturbation (most commonly an expansion of the recognition template, so that a scheme applies more broadly, von Glasersfeld, 1995). Typically accommodations involve a curtailing that is the reverse of the most common form of assimilation. Processes of assimilation and accommodation are thus intertwined, as assimilations lead to accommodations that lead to assimilations (von Glasersfeld, 1995).

Schemes can be thought of as recursive in the sense that the output of a scheme can become part of the activity of another scheme. I use the term *meta-scheme* to refer to processes that act on schemes (cf., Piaget, 1970). I thus consider the processes of assimilation and accommodation as meta-schemes. Schemes for internalization and interiorization of schemes are also meta-schemes whose input is a scheme itself. For the vast majority of situations, meta-schemes are enacted without meta-cognitive awareness, but learners also develop a meta-scheme of *reflective abstraction*. Reflective abstraction begins with reasoning about prior experiences, via *re-presentations* (mental recordings of prior experiences). Via processes of abduction, perturbation, assimilation and accommodation, these re-presentations can become successively more abstract. Reflective abstraction is thus an accommodation of an individual's meta-schemes to include more awareness, control, and flexibility. Due to limitations of working memory, reasoning about successively more abstract re-presentations of mental objects both requires and necessitates interiorizations of systems of mental actions on those objects as *conceptual structures*, which are systems of interiorized operations (Piaget, 1970; Norton & Bell, 2017).

Units Coordinating Structures

A units coordinating structure defines and regulates relationships between transformed *units* as possible, logically necessary, and reversible (Boyce & Norton, 2017). Here a *unit* refers to a size, and transformations include operations of partitioning and iterating as well as composing (putting one unit inside another unit) and disembedding (removing a copy of a unit from within a composite unit without modifying the composite unit). Such operations are constructed by students as part of their process of constructing sequences of counting numbers and reorganized to apply to fractions (Norton & Wilkins, 2012) and integers (Ulrich, 2015).

Individuals' schemes for rational number are thus characterized in part by their levels of units (Steffe & Olive, 2010), where the number of levels of the structure refer to the nestedness of reversible coordinations. The iterative fraction scheme requires assimilation with a units coordinating structure relating three levels of units (e.g., four 1/4 units within one and nine 1/4 units within 9/4). Assimilation of fractional situations with three levels of units allows a student who has constructed an interiorized iterative fraction scheme to *anticipate* iteration of an amount determined by partitioning before actually carrying out the partitioning with internalized or physical objects.

Need for Distinguishment of Covariational Reasoning Levels by Pseudo-Empirical, Internalized, or Interiorized Mental Activity

Note that with one exception, the descriptions of the covariational reasoning levels (depicted in Figure 1) refer to forming mental imagery, which I associate with internalized schemes. I propose theorized distinguishment of pseudo-empirical and interiorized covariational reasoning. I contend that to understand learners' development of covariational reasoning across levels requires understanding their pseudo-empirical, internalized, and interiorized reasoning within levels. For instance, whereas internalized continuous covariational reasoning (both smooth and chunky) might require assimilation with three levels of units, perhaps students assimilating with two levels of units can construct pseudo-empirical schemes for smooth continuous covariational reasoning. Consideration of this additional lens can help to inform the field of more specific learning trajectories and support the design of tasks engendering *perturbation*, *abduction*, *assimilation*, *accommodation*, and *reflective abstraction* that result in students' construction of more powerful covariational reasoning.

Framing mental imagery associated with levels of covariational reasoning as internalized activity of covariational reasoning schemes allows for other representations of internalized actions (such as imagery of zooming in or out on the graph of an emergent trace (Ellis, Ely, Singleton, & Tasova, 2018) that may require the same levels of units interiorized. More generally, it allows for identifying learning trajectories within and across levels of covariational reasoning that extend beyond descriptions of internalized mental activities to include focus on how students act upon standard and non-standard representations of graphs (Frank, 2018; Paoletti & Moore, 2017) and equations (Stevens, 2019) as part of analyses of covariational reasoning.

Level	**Description from (Thompson & Carlson, 2017, p. 440)**	**Proposed Distinguishment by Pseudo-empirical, Internalized, or Interiorized Reasoning**
Smooth continuous covariation	The person envisions increases or decreases (hereafter, changes) in one quantity's or variable's value (hereafter, variable) as happening simultaneously with changes in another variable's value, and the person envisions both variables varying smoothly and continuously	Interiorized: The person anticipates smooth and continuous covariation between two quantities without necessarily forming mental imagery. Pseudo-empirical: The person evokes reasoning about a smooth and continuous representation without envisioning covariation between two quantities.
Chunky continuous covariation	The person envisions changes in one variable's value as happening simultaneously with changes in another variable's value, and they envision both variables varying with chunky continuous variation.	Interiorized: The person anticipates chunky and continuous covariation between two quantities without necessarily forming mental imagery. Pseudo-empirical: The person evokes reasoning about a chunky and continuous representation without envisioning covariation between two quantities.
Coordination of values	The person coordinates the values of one variable (x) with values of another variable (y) with the anticipation of creating a discrete collection of pairs (x, y).	Interiorized: The person anticipates correspondence between two variables' values without necessarily forming mental imagery of their pairing. Pseudo-empirical: The person anticipates forming a new representation of discrete correspondences by which to reason about changes.

Gross coordination of values	The person forms a gross image of quantities' values varying together, such as "this quantity increases while that quantity decreases." The person does not envision that individual values of quantities go together. Instead, the person envisions a loose, nonmultiplicative link between the overall changes in two quantities' values.	Interiorized: The person anticipates binary correspondences between two variables' changes without forming mental imagery. Pseudo-empirical: The person identifies and reasons about representations of binary correspondences between two variables' changes.
Pre-coordination of values	The person envisions two variables' values varying, but asynchronously—one variable changes, then the second variable changes, then the first, and so on. The person does not anticipate creating pairs of values as multiplicative objects.	Interiorized: The person anticipates an asynchronous sequence of binary changes in values without forming mental imagery. Pseudo-empirical: The person identifies and reasons about representations of an asynchronous sequence of binary changes in values.
No coordination	The person has no image of variables varying together. The person focuses on one or another variable's variation with no coordination of values.	Internalized: The person forms an image of one variable varying. Pseudo-empirical: The person reasons about variation in one variable by relying on a representation.

Figure 1. Covariational Reasoning Level Descriptions

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