DISCOVERING SQUARE ROOTS: PRODUCTIVE STRUGGLE IN MIDDLE SCHOOL MATHEMATICS

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Using videos and transcripts from a lesson on square roots and follow-up conversations with students and teachers, we analyze factors that can facilitate episodes of productive struggle, including student and teacher dispositions, task features, and classroom conditions. We highlight choices and task features that honored student autonomy and maintained students' engagement in, and success with, the lesson. We also discuss students' equitable access to the mathematics of the lesson.

Keywords: Problem Solving, instructional activities and practices, and communication.

Background and Theoretical Framework

The phenomenon of productive struggle is a relatively new research focus in the study of classroom mathematical activity. Warshauer (2015) characterized productive struggle in terms of students' "effort to make sense of mathematics, to figure something out that is not immediately apparent" (Hiebert & Grouws, 2007, p. 287). Her findings suggest that teacher interactions with students in moments of productive struggle can either maintain or subvert the cognitive demands of tasks, thus strengthening or hindering student learning and understanding.

Sengupta-Irving and Agarwal (2017) discuss the implications of collective effort and decision making in persistence through challenging episodes, as well as indicators of these occurrences that allow educators to facilitate productive peer interactions. Granberg (2016) investigates the role of student observation and analysis of errors and mistakes in turning struggle into a productive event. She relates her definition of unproductive struggle to Schoenfeld's (1985) description of the behavior of novices, in which ideas are not revisited and knowledge is not reconstructed. Zeybek claims that tasks with high-level cognitive demands, specifically those which lend themselves to multiple approaches and allow for more than one correct answer, are essential to students' development of deep understanding (Zeybek, 2016).

In keeping with Harel's Necessity Principle (Harel, 2013), we hypothesize that students' acquisition of mathematical ideas is both more likely and more robust when animated by an intellectual need to understand a situation or solve a problem. Therefore, we conceptualize productive struggle as follows: we say that students are engaged in *productive struggle* when they autonomously attempt to use resources - including their own knowledge, knowledge of their peers and teachers, and physical resources such as technological tools - to overcome an intellectual obstacle, and when this process leads to the discovery or consolidation of a mathematical idea, technique, or problem-solving strategy.

Guided by this framework, we aim to address the question: *What student and teacher dispositions, task features, and classroom conditions are conducive to productive struggle?*

Study Method and Participants

This study was conducted at a two-week summer mathematics program for upper-elementary and middle school students in 2018. Our studied focused on one course which covered concepts of area, perimeter, and the Pythagorean theorem. The course was taught by an inservice seventh grade algebra teacher (Rita) and included seventeen students entering grades 6 and 7, two preservice

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teachers (PSTs), and a doctoral student with experience teaching in special education programs. Sixteen students, Rita, and her three assistants (the two PSTs and the graduate student) participated in our study. Eight students were male and eight were female; all but one student belonged to ethnic minority groups.

The classroom was videotaped at all times, with a second camera used to capture episodes of small-group work. During breaks, we often conducted brief audiotaped interviews with Rita to capture her perspective on activity that had just occurred. After each day's class, Rita and her assistants completed written reflections and participated in a videotaped small-group debrief of the morning's events.

Throughout the program, we observed and recorded instances in which the entire class's activity met our criteria for productive struggle: that students are engaged in a sustained effort to overcome an intellectual obstacle; that students autonomously use resources to overcome the obstacle; and that the effort results in the discovery or consolidation of a mathematical idea, technique, or strategy. In this report, we present our analysis of one such instance, informed by observations from both students and teachers, in order to illuminate characteristics of a lesson and learning environment that can stimulate productive struggle.

Data Analysis

Our analysis of classroom, debrief, and interview transcripts focuses on an episode that occurred during the seventh day of the program. Following an activity on the Pythagorean theorem, Rita presented students with a planned sequence of problems on squares, square roots, and areas and perimeters of rectangles. Each time Rita presented a problem, students were asked to work out the problem on individual whiteboards at their seats, concealing their work until Rita prompted the entire class to reveal their answers. Students' work on the problems revealed some lingering confusion about the distinction between squaring a number and taking the square root of a number and the distinction between the area and the perimeter of a square; Rita and her assistants addressed these confusions as the class progressed through the tasks.

For the last problem in the sequence, Rita drew a picture of a square, labeled the area as "A = 40 cm^2", and asked students for the length of a side ("l = ?"). Videos of students' work revealed that most students approached this problem by observing that the length must be between 6 and 7 centimeters, then attempting to obtain increasingly precise approximations for the side length by iteratively selecting decimal numbers between 6 and 7, squaring them by hand, and using the result to decide whether to guess a higher or lower value for the side length.

Rita led a brief discussion of side lengths that students had tried so far. This discussion organically reverted to small group and individual work, with students attempting to refine approximations. During this process, a student discovered a bag of four-function calculators (with square root keys) behind the teacher's desk, and Rita encouraged students to use them to continue their work on the problem. The episode closed with a discussion of how the square root function on the calculator could quickly provide an approximation for the side length of the square, and why no terminating decimal value would yield an area of exactly 40.

We selected four students for one-on-one interviews at the end of the lesson based on the key roles they played in the development of the episode. During interviews, students had access to rulers, four-function calculators with square root keys, grid paper, and sheets of blank copy paper, as well as their personal electronic devices, when they were available.

In reviewing classroom and interview data, we conducted a thematic analysis (Braun & Clarke, 2006), coding themes in classroom discourse and in students' and teachers' descriptions of the episode that corresponded to elements of our framework for productive struggle or suggested factors that contributed to the episode's development. We organized our analysis according to the extended instructional triangle of Boerst and Ball (2007), which suggests we may gain insight by analyzing the work of students, work of teachers, mathematics in which students and teachers are engaged, and learning environment. We analyzed video and transcripts of the classroom episode, and then triangulated our observations with insights from our interview with Rita and the post-class debrief. We summarize the themes we identified in Table 1.

Table 1: Factors Supporting Productive Struggle in the Square of Area 40 Task

Student Factors	**Teacher Factors**
Curiosity about the value of an unknown quantity* Willingness to persist through lengthy calculations* Disposition to use tools strategically*+ Approximation sense: understanding when a guess is too high or too low; deciding when to move closer to one endpoint of an interval	Commitment to growth mindset (shared by Rita and assistants)* Commitment to probing rather than directing student thinking Disposition to enlist students in addressing each other's thinking Disposition to allow students to use technology to make work more efficient* Strategies for sustaining student motivation and engagement
Task Factors	**Learning Environment Factors**
Problem was "not too easy, not too hard"+ Problem came at the end of a sequence of tasks that gradually increased in difficulty* Problem addressed students' conceptions of perimeter and area* Problem rewarded sustained effort with successively more accurate approximations	Calculators were available in room, but not immediately visible* Classroom norms permitted students to move about the room and share ideas and approximations with each other
* Indicates an observation supported by teachers' comments during the interview or debrief + Indicates an observation supported by students' comments during post-task interviews	

Based on our observations as well as those of Rita and the assistants, the children's work on the Square of Area 40 task satisfied our criteria for productive struggle. Their work was productive in that it created learning opportunities that were realized in the collaboration among students and teachers: the co-construction of a process for interpolating a solution x to the equation $x^2 = a$, where a is a given real number, the consolidation of this process into the notion of the square root, and a discussion of how a calculator rounds the results of calculations and can appear to give a whole number output, even when the true value is not a whole number.

Teachers' discussions after the lesson revealed several student dispositions that contributed to the lesson's success in stimulating productive struggle. Foremost among these was genuine curiosity and excitement about the problem situation, as the doctoral student pointed out during the post-class debrief: "[the] question created so much engagement … intense discussion, action, like everybody's running around, digging in to get that exact number." Another factor was the students' willingness to persist through laborious pencil-and-paper calculations as they tested their guesses. During the episode, several students tested guesses that had three significant digits of precision. In the post-class debrief, the doctoral student recalled that "they were just … very excited, trying to be precise." Also important, in our own observation, was the students' strategic knowledge for approximation. For example, at least one student observed that since 40 was closer to 36 than to 49, her first guess for the side length should be closer to 6 than to 7. Students' approximation sense helped them obtain more accurate guesses for the square's side length, allowing them to obtain areas closer to 40 and bolstering their motivation to continue.

Both Rita and one of the PSTs spoke about the importance of helping students interpret struggle and temporary adversity through a growth mindset, so that students were not discouraged by temporary setbacks. Several choices by Rita and her colleagues bolstered students' sense of mathematical agency, and their disposition to encourage students to help one another through obstacles. This included heavy use of probing questions and tendencies to frequently have students describe their work in detail. We hypothesize that such facilitation moves by teachers can help students reinforce the conceptual content of a lesson for themselves and others, and allows them to enlist their peers as resources in pursuing a shared strategy.

Characteristics of the classroom environment also played a role in the episode's development. In their initial work on the task, students performed calculations by hand. Gradually, some students began to use the calculator functions of their phones to explore more efficiently; some students noticed this and questioned whether the use of calculators was contrary to class norms. The initial lack of calculators encouraged students to perform calculations by hand, which helped them become accustomed to the parameters and goals of the problem. However, the availability of calculators to only students with personal devices threatened to limit some students' access to the full depth of the task. The class set of calculators helped to resolve this, though some students still had access to more precise approximations.

Technology played an additional role in the resolution of the task: one student discovered that when she squared one of her estimates on her phone, the device reported that the square was 40. However, switching the phone to landscape orientation revealed more digits, showing that the value was equal to 40 to only thirteen decimal places. This helped the student to see that the device was automatically rounding values, even when it appeared to report a whole-number answer. Rita, noticing this insight, asked the student to share it with the entire class, again helping one student's access to the mathematics in the task become a resource for others.

Discussion

Granberg (2016) posits that when students are prematurely offered procedures for solving mathematics problems, the intellectual challenge of problem solving is lost, and students' inquiry into the nature of such problems is cut short. The Square of Area 40 task led to an episode of productive struggle in which students refined their conceptions of squares and square roots in an exploratory way without being guided to use a particular procedure or strategy. We hypothesize that the autonomy afforded to students during this activity strengthened their sense of agency as well as their understanding of the nature and purpose of the task. This sense of autonomy was a result of choices made by both students and teachers during the activity.

We hypothesize that the conditions we have described - student and teacher dispositions, task features, and classroom conditions - can be tailored to create a learning environment which facilitates productive struggle. These factors can further be considered in bringing productive struggle from a thing we strive to achieve into something that can be operationalized in creation of tasks to be facilitated in the research environment. However, some care is necessary to ensure that the benefits of productive struggle accrue to all students and that the knowledge and resources enjoyed by some students are marshalled for the benefit of the entire class. In orchestrating the lesson, Rita and her colleagues explicitly reinforced productive processing of adversity, encouraging students when they made mistakes or when their calculations did not turn out as anticipated. They also noticed some of the insights available to students with personal handheld devices and worked to draw other students into discussions about these insights. By attending to issues of equity, the teachers enhanced the lesson's effectiveness and deepened students' opportunities for learning.

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