STUDENTS' UNDERSTANDING OF LINEAR ALGEBRA CONCEPTS UNDERLYING A PROCEDURE IN A QUANTUM MECHANICS TASK

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As undergraduate physics students solve problems related to quantum mechanics, they often have to draw on their conceptual understanding of both linear algebra and physics concepts. One concept that is prevalent in both of these disciplines is change of basis. In this study, we focus our analysis on physics students' mathematical conceptual knowledge of principles that underlies their use of a procedure on a quantum mechanics task. As students perform this task, they demonstrated understanding of change of basis and properties of orthonormal bases. We share examples of students' reasons for performing a change of basis and demonstrate how students seem to draw on their conceptual understanding of properties of orthonormal bases as they implement the procedure. We thus exemplify how physics students apply their mathematical knowledge in interdisciplinary contexts.

Keywords: Interdisciplinary studies, STEM / STEAM, University mathematics

Undergraduate physics students are commonly required to take a course in linear algebra, seen as several concepts and procedures covered in a linear algebra course are also addressed in physics courses, particularly quantum mechanics. Physics students have to make connections between concepts covered in both linear algebra and physics contexts. This may be challenging for students, given that concepts included in both linear algebra and quantum mechanics courses are sometimes used differently in the two different contexts. One such example is that bases are typically orthonormal in quantum mechanics contexts (McIntyre, Manogue, & Tate, 2012), which is generally not the case in linear algebra. Furthermore, a change of basis in a quantum mechanics context may be performed differently than a change of basis in a linear algebra context, such as through formulaic substitution of commonly used bases rather than a change of basis matrix. Since instructors want students to have a cohesive understanding of basis and change of basis across these interdisciplinary contexts, it is useful to explore how students reason about change of basis in a quantum mechanics context.

As students solve problems related to quantum physics, they often have to draw on their conceptual understanding of both mathematics and physics. In this study, we focus our analysis on physics students' mathematical conceptual knowledge that underlies their use of a procedure on a quantum mechanics problem. We address the following research question: How do students use their mathematical conceptual knowledge as they perform a quantum mechanics task?

Theoretical Framing

Conceptual and procedural knowledge are constructs commonly used by researchers to characterize students' understanding of mathematical concepts. Rittle-Johnson, Schneider, and Star (2015) have theorized that bidirectional relations exist between students' conceptual knowledge and procedural knowledge. These researchers asserted that procedural knowledge can support students' development of conceptual knowledge and vice versa. Particularly, conceptual knowledge can support students' flexibility in choosing appropriate procedures (e.g., Baroody, Feil, & Johnson, 2007). Crooks and Alibali (2014) identified two main types of conceptual knowledge, general principle knowledge and knowledge of principles underlying procedures. *Knowledge of principles underlying procedures* involves understanding the "connections among the steps in a procedure and between individual

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steps and their conceptual underpinnings" (p. 367). In this study, we use Crooks and Alibali's (2014) construct of knowledge of principles underlying procedures to frame our analysis of students' conceptual knowledge of the mathematical concepts underlying their use of a procedure in quantum mechanics problem.

Brief Physics Background

Quantum mechanical systems and all knowable information about them are represented mathematically by normalized kets, symbolized in Dirac notation as $|\psi\rangle$. Kets mathematically behave like vectors, and a ket's complex conjugate transpose, called a bra, is symbolized as $\langle\psi|$. Spin is a measure of a particle's intrinsic angular momentum and is represented mathematically by an operator such as $\hat{S}_z$ (where the z indicates the particle's axis of rotation). In a spin-½ system, there are two possible results for the S_z measurement: $\pm\frac{\hbar}{2}$; they correspond to $|+\rangle$ and $|-\rangle$, which comprise a set of orthonormal basis vectors called the S_z basis. Any quantum state $|\psi\rangle$ is a linear combination of them: $|\psi\rangle = a|+\rangle + b|-\rangle$. In this study, we analyze the students' responses to the two interview questions presented in Figure 1. Interview question (a) asks students to determine the probability of obtaining $\frac{\hbar}{2}$ or $\frac{-\hbar}{2}$ in a measurement of the observable S_z on a system in state $|\psi\rangle$. This is calculated by $P_\pm = |\langle\pm|\psi\rangle|^2$, where $\langle\pm|\psi\rangle$ is an inner product between one of the basis kets and psi. Because $|\psi\rangle$ is written as a linear combination of the two vectors that comprise the z-basis, solving this problem requires no change of basis. The analogous information can be determined for other axes of rotation, such as y. To complete question (b), a change of basis is involved because the given state vector $|\psi\rangle$ is written in terms of the z-basis, but the prompt asks for the probability that the spin component is up along the y-axis. The two main approaches are to either change $|\psi\rangle$ to be written in terms of the y-basis (denoted $|\pm\rangle_y$), or change the y-basis vectors to be written in terms of the z-basis. In either change of basis approach, one would need to utilize the relations, $|\pm\rangle_y = \frac{1}{\sqrt{2}}|+\rangle \pm \frac{i}{\sqrt{2}}|-\rangle$.

Consider the quantum state vector $|\psi\rangle = \frac{3}{\sqrt{13}}|+\rangle + \frac{2i}{\sqrt{13}}|-\rangle$.

Calculate the probabilities that the spin component is up or down along the z-axis.

Calculate the probabilities that the spin component is up or down along the y-axis.

Figure 1: The interview questions analyzed in this paper.

Methods

Semi-structured interviews (Bernard, 1988) were conducted with 12 quantum physics students, of which eight were enrolled in a junior-level course at a large public research university (A) in the northwest United States, and four were enrolled in a senior-level course at a medium public research university (C) in the northeast United States. Students from these two universities were assigned pseudonyms of A# and C#. Both courses used a "spins first" approach, with McIntyre et al. (2012) as their course textbook. The interview questions were designed to elicit evidence of student understanding of linear algebra concepts used in their quantum mechanics course. We analyzed the students' responses to the two interview questions presented in Figure 1. A relevant follow-up question of particular interest was: "How do you see this problem relating to basis or change of basis?"

The data collected from these interviews include video recordings of the interviews and copies of student work. We transcribed the students' responses to these tasks and wrote descriptions of the procedures the students used and the mathematical concepts the students discussed as they explained their thought process and justified their choice of procedure. In writing these descriptions of the students' interview responses, we noticed the students commonly referred to mathematical concepts of orthonormal bases, inner product, and change of basis. We thus focused our qualitative analysis on students' conceptual understanding of these mathematical concepts underlying the probability procedure in this quantum physics context. We performed open coding (Miles, Huberman, & Saldaña, 2013) of the procedures the students used and the mathematical concepts they demonstrated an understanding of in their responses to the interview questions. We then wrote analytic memos (Maxwell, 2013) reflecting on how the students' understanding of linear algebra concepts seemed to support their use of the procedure.

Results

We present our results in two subsections. We first discuss the procedures the students employed to solve task. We then discuss the nature of the conceptual understanding the students exhibited as they performed this procedure and justified their procedure choice.

Students' Procedures for the Task Requiring a Change of Basis

In task (b), the students could either change $|\psi\rangle = \frac{3}{\sqrt{13}}|+\rangle + \frac{2i}{\sqrt{13}}|-\rangle$ to be written in terms of the y basis or change $|+\rangle_y$ (and $|-\rangle_y$for spin down) to be written in terms of the z-basis. Three out of twelve students used the former approach, but only one did so correctly. This student, A8, added the change of basis equations (given on a reference sheet) $|+\rangle_y = \frac{1}{\sqrt{2}}|+\rangle + i\frac{1}{\sqrt{2}}|-\rangle$ and $|-\rangle_y = \frac{1}{\sqrt{2}}|+\rangle - i\frac{1}{\sqrt{2}}|-\rangle$ to find $|+\rangle_y + |-\rangle_y = \frac{2}{\sqrt{2}}|+\rangle$. He then multiplied both sides of the equations by $\frac{\sqrt{2}}{2}$ to find $\frac{1}{\sqrt{2}}|+\rangle_y + \frac{1}{\sqrt{2}}|-\rangle_y = |+\rangle$, which is a z-basis vector written as a linear combination of the y-basis vectors. He then subtracted the given change of basis equations to find $|+\rangle_y - |-\rangle_y = \frac{2}{\sqrt{2}}i|-\rangle$ and multiplied both sides by $\frac{\sqrt{2}}{2i}$ to find $\frac{1}{\sqrt{2}i}|+\rangle_y - \frac{1}{\sqrt{2}i}|-\rangle_y = |-\rangle$. He then substituted $|+\rangle = \frac{1}{\sqrt{2}}|+\rangle_y + \frac{1}{\sqrt{2}}|-\rangle_y$ and $|-\rangle = \frac{1}{\sqrt{2}i}|+\rangle_y - \frac{1}{\sqrt{2}i}|-\rangle_y$ into $|\psi\rangle = \frac{3}{\sqrt{13}}|+\rangle + \frac{2i}{\sqrt{13}}|-\rangle$ and simplified the equation to find $|\psi\rangle_y = \frac{5}{\sqrt{26}}|+\rangle_y + \frac{1}{\sqrt{26}}|-\rangle_y$. To calculate the probability that the spin component is up along the y-axis, he computed $\left|\,{}_y\langle+|\psi\rangle_y\right|^2 = \frac{25}{26}$ by squaring the coefficient of $|+\rangle_y$, taking for granted the fact that ${}_y\langle+|+\rangle_y = 1$ and ${}_y\langle+|-\rangle_y = 0$ because the basis vectors are orthonormal.

For the calculation in question (b), ten students changed $|+\rangle_y$ to be written in terms of the z- basis, and all of them used the appropriate change of basis formula $|+\rangle_y = \frac{1}{\sqrt{2}}|+\rangle + i\frac{1}{\sqrt{2}}|-\rangle$. Seven of the ten students who performed the rest of the procedure found the conjugate transpose as ${}_y\langle+| = \frac{1}{\sqrt{2}}\langle+| - \frac{1}{\sqrt{2}}i\langle-|$ and substituted this into the probability formula $P_{+,y} = \left|\,{}_y\langle+|\psi\rangle\right|^2$ to find $P_{+,y} = \left|\left(\frac{1}{\sqrt{2}}\langle+| - \frac{1}{\sqrt{2}}i\langle-|\right)\left(\frac{3}{\sqrt{13}}|+\rangle + \frac{2i}{\sqrt{13}}|-\rangle\right)\right|^2$. They then distributed and used the orthonormality properties, $\langle+|+\rangle = 1$, $\langle-|-\rangle = 1$, and $\langle+|-\rangle = 0$ to simplify the equation to be $P_{+,y} = \left|\frac{5}{\sqrt{26}}\right|^2$, which gave a probability of $\frac{25}{26}$.

Students' Conceptual Knowledge of Linear Algebra Underlying These Procedures

As the students performed this procedure, they demonstrated conceptual knowledge of the linear algebra concepts of change of basis and properties of orthonormal bases. All of the students recognized a need to change the basis for this problem. They often explicitly acknowledged that they could not calculate this inner product without performing a change of basis. For instance, A13 claimed, "you can't just multiply out in, when they're [the vectors are] in different bases." As C6 described how this problem was related to change of basis, she claimed, "you can't do anything until you're in the same basis." The students each demonstrated conceptual understanding of why a change of basis was necessary in this task. Other students noted that performing the change of basis made the calculation simpler. For instance, in A8's change of basis procedure described above, changing the basis of $|\psi\rangle$ to express the ket as a linear combination of y-basis kets allowed him to use the simple procedure of squaring the norms of coefficients of y-basis kets. Students who used the second procedure discussed how changing basis makes the calculations simpler because they can use orthonormality property of the bases.

The students' motivation for selecting the procedure of changing the basis was the ability to take advantage of these properties of an orthonormal basis. A21 claimed it was necessary to change basis in order to make assumptions about $\langle+|-\rangle = 0$. C5 also suggested that a change of basis was necessary for the "inner products to be nice." The students demonstrated conceptual understanding of properties of orthonormal bases, which underlie their procedures for this task. Their procedures were particularly dependent on the fact that the y-basis and the z-basis are both orthonormal, which implies that $\langle+|+\rangle = 1$, $\langle-|-\rangle = 1$, and $\langle+|-\rangle = 0$.

Discussion

As the students used the change of basis procedure in this quantum mechanics problem, they demonstrated conceptual understanding of the orthogonality and normality properties of basis vectors and the associated inner product relations of $\langle+|+\rangle = 1, \langle-|-\rangle = 1,$ and $\langle+|-\rangle = 0$. The students seemed to draw on their understanding of the principles underlying this procedure as they performed the probability calculation and justified their choice of procedure. Their conceptual understanding of these mathematical properties seemed to support them in their choice of procedure and their implementation of it. Baroody et al. (2007) suggested that students' conceptual knowledge can supports students' flexibility in applying procedures. Our study further illustrates how students' conceptual understanding of linear algebra concepts can be useful in supporting their flexibility in performing procedures in quantum mechanics contexts.

In this quantum mechanics problem, the students' change of basis approaches involved employing several mathematical properties. However, their method for performing the change of basis though algebraic substitution is different from how they would perform a change of basis in a linear algebra course, which typically involves the use of a change of basis matrix. Physics students may experience challenges in using their understanding of linear algebra to solve problems in quantum mechanics contexts. Therefore, we suggest that future research can address how physics students transfer their understanding of change of basis across linear algebra and quantum mechanics contexts. Researchers can particularly focus on how physics students resolve differences they experience as they make connections across these disciplines.

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