CONSTRUCTING RATES OF CHANGE THROUGH A UNITS COORDINATING LENS: THE STORY OF RICK

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In this research report, we describe the results of a paired-student constructivist teaching experiment with introductory calculus students focused on supporting their understanding of the derivative as rate of change. We focus on one student, Rick. We connect analyses of Rick's ways of assimilating and operating with numerical units with analyses of ways of conceptualizing rates. The results are conjectures about the relationships between levels of units students coordinate and their ways of quantifying rates.

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Background

This report builds a connection between Johnson's (2015) work investigating the quantitative operations involved in constructing rates and research investigating the constraints and affordances of student's units coordinating activity as they develop understandings of calculus concepts (Boyce, Byerley, Darling, Grabhorn, & Tyburski, 2019; Boyce, Grabhorn, & Byerley, 2020). The goal of the current study is to identify connections between calculus students' units coordination and their understanding of rate of change.

Units Coordinating

Units coordination can be thought of as students' mental activity building and maintaining relationships of nested *levels* of units (Norton et al., 2015; Steffe, 1992). Some students bring a three-level units coordinating structure to bear when first encountering a task, what we call *assimilating* with three levels of units. Such students would be able to quickly reason through the Bar Task below (Figure 1) by recognizing that the orange bar is equivalent to nine ¼'s of a purple bar, thus 9/4 of a purple bar fits into the orange bar. Students that assimilate with two levels of units may construct an ephemeral third level of units in the midst of reasoning (what we call coordinating three levels of units *in activity*) by coordinating across two two-level units coordinating structures. Such activity requires perceptual reflecting on the outcomes of actions on physical or mental representations, often resulting in conflating or dropping units. For example, a student who assimilates with two levels of units may state 2 1/9 as an answer to the Bar Task by claiming that two full purple bars and one green bar (1/9 of an orange bar) fit into the orange bar.

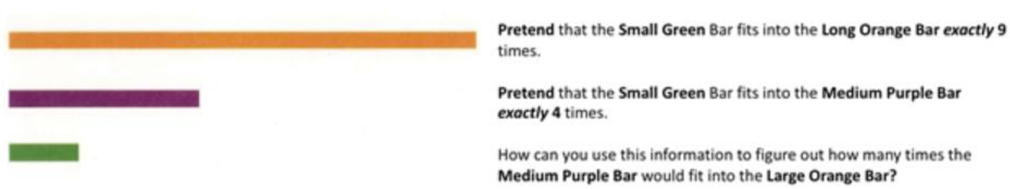

Figure 1: Units Coordinating Bar Task (Norton et al., 2015)

Rate of Change as a Ratio

Johnson (2015) investigated the affordances and constraints of secondary students' quantification of ratios in regard to their quantification of rate. The resulting Change in Covarying Quantities Framework distinguishes between quantitative operations involved in students' quantification of rate:

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comparison refers to a quantification of rate as associations of changes of quantities (i.e., 12 miles per hour means associating a distance of 12 mile with an elapsed time of 1 hour), while *coordination* refers to quantification of rate as involving at least one continuously changing quantity (i.e. 12 miles per hour means as time increases, the distance traveled is 12 times as large). Johnson argued that "students' quantification of rate could help to explicate differences in students' conceptions of rate" (p. 86-87) and that the nature of how students might develop either operation of *comparison* and *coordination* is unanswered. We hypothesize that students' units coordinating activity may be helpful for understanding the nature of such operations.

Methods and Results

We conducted a paired-student teaching experiment (Steffe & Thompson, 2000) in summer 2019 at a large public university in the U.S. with the goal of producing models of introductory calculus students' developing understandings of rate of change. Our results and analysis will focus on one participant, Rick. Rick participated in five weekly one-hour teaching episodes concurrent with his enrollment in a differential calculus course. The first author served as teacher-researcher for each episode while the second author served as a witness. Each episode was video recorded and all written work was collected and scanned for analysis. Analysis methods included both ongoing (between session) and retrospective modeling of Rick's ways of coordinating units and Rick's ways of reasoning about rates of change (Steffe & Thompson, 2000). Rick was assessed as assimilating with two levels of units at the beginning of the teaching experiment. The following analysis details our attempt to support Rick in quantifying a rate via the *coordination* operation of Johnson's (2015) Change in Covarying Quantities Framework.

12 Meters In 3 Seconds

During the third teaching episode we focused on supporting Rick in quantifying a rate with the *coordination* operation. Rick's conception of a rate as the amount of change in a dependent quantity for a unit increase in an independent quantity was persistent. Rick was presented with the task displayed in Figure 2.

Compare and contrast the following statements.

A. I travel 12 meters in 3 seconds.
B. I travel at a constant rate of 12 meters per 3 seconds.
C. I travel at a constant rate of 12/3 meters per second.
D. It takes me ¼ seconds to travel 1 meter.

Figure 2: Comparing Rates Task

Rick first noted that statements B and C are similar because they were "a constant rate… over a certain interval of time", but that those statements are different from statements A and D because the latter pair did not reference a constant rate. When pushed to describe other similarities or differences, Rick claimed that "throughout the board… each second they would have traveled 4 meters". Even though Rick attended to the absence or inclusion of the phrase *constant rate*, he still compared statements by considering the amount of distance covered in 1 second (as if each statement referred to a constant rate of change between distance and time). Rick then stared at option A and claimed "actually, I don't know that". He then explained that each statement described traveling 12 meters in 3 seconds. The teacher-researcher then asked Rick to give an example where statement A is true, but does not describe traveling 4 meters in 1 second.

> Rick: Potentially within the first second maybe you're, uh… stopped the entire time. And then, so zero to one [seconds] you travel zero [meters]. Then one to two seconds you travel six [meters]. Then two to three [seconds] you travel another six [meters]. So, in one second it's not guaranteed to be four [meters] in that particular situation.

Rick was able to give several additional examples by considering individual changes in distance across three successive elapsed seconds, where the sum of changes in distance was 12 meters. Additionally, some of his examples included traveling four meters in one of the elapsed seconds but not all (e.g., 0 meters in the first second, 8 meters in the next second, 4 meters in the final second). According to Rick, "it could be any mixture of numbers leading to twelve".

We interpret Rick's response as indicating that, to him, a constant rate of 4 meters per second means that distance must change by four meters for *every* second he can consider throughout the trip. Additionally, Rick interprets any constant rate described with a non-unit change in independent quantity (in this case, time) by finding the associated change in dependent quantity per increase of one unit of the independent quantity that would maintain the originally stated ratio.

This explains why Rick singled out choice A as different than choices B and C but not D; For choice D, distance must change by four meters if we consider iterating ¼ second four times to let one second elapse. Even though Rick had an awareness that amounts of changes in distance can vary as time elapses for both statements A and D, Rick's image of that variation necessitated considering unit increases in time. Rick did not consider the variation of distance for individual ¼ second intervals of time, and thus statement D is consistent with statements B and C (in these three cases, for Rick, distance *must* increase by four meters for any increase of one second).

12 Meters In 0.8 Seconds

The previous task revealed that Rick could reason about rate of change by considering amounts of change in a dependent quantity constrained by an associated increase of the independent quantity by one unit. The following excerpt describes Rick attempt to reason about a rate with a non-unit change in the independent quantity. Specifically, we ask Rick to compare a statement similar to the previous task with a statement about instantaneous rate of change.

> Teacher-Researcher: Jim travels 12 meters in 0.8 seconds. Is it possible Jim traveled 12 meters per second at any point during his trip?
>
> Rick: Potentially… I don't want to say… Because I was thinking potentially Jim could go… travel 15 meters per second but stop at, you know, 0.8 seconds. And then Jim would be traveling… no… If you travel 15 [meters]… if Jim would stop at 0.8 [seconds] exactly after having traveled that, then 12 [meters] is eighty percent of 15 [meters].

Rick's activity is focused on relating a change in distance of 12 meters to a unit rate that describes traveling 12 meters in 0.8 seconds. One interpretation of his response is that Rick interprets "12 meters per second" as traveling 12 meters *within* one second, and thus it is possible to travel 12 meters per second during the trip.

Rick constructs (at least) three rates as he attempts to solve this task: 12 meters per 0.8 seconds (Figure 3a), 15 meters per second (Figure 3b), and 12 meters per second (Figure 3c). In each case, Rick can reason about a rate as a comparison of changes in distance and changes in time and appears to prefer reasoning about such rates over a unit interval (one second) of time. This may be due to Rick having not interiorized a conceptual structure for non-unit rates, thus necessitating activity to construct a unit rate with which to reason.

Rick can compare two speeds by constructing unit rates for speed and comparing the two changes in distance associated with a common unit increase in time. In doing so, Rick does not attend to time as a quantity that is necessary for his goal of comparing two speeds. This may explain his initial response that Jim could potentially travel 12 meters per second by traveling "15 meters per second but stop at, you know, point-eight seconds". Rick's suggestion links a unit rate with a change in distance of 12 meters.

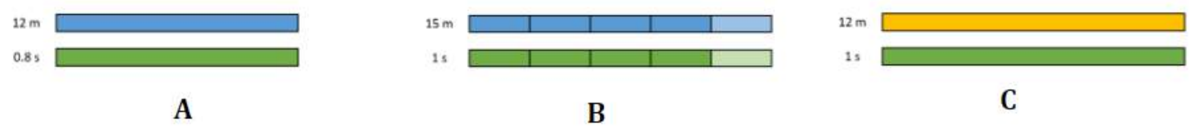

Figure 3: Author illustrations of rates that Rick constructs and compares

Ultimately Rick decided that it was not possible that Jim traveled 12 meters per second at any point during his trip because, "if we're going that specific interval [one second], that'd be fifteen meters per second. Not twelve".

Discussion

The goal of our research was to investigate how Rick, who we assessed as assimilating with two levels of units, reasoned about rate of change. We have focused analysis on two particular tasks to exemplify the powerful ways that Rick was able to reason about unit rates and able to coordinate three levels of units in activity involving known quantities. Still, throughout the teaching experiment, Rick did not exhibit behavior indicating that he constructed rates by engaging in the quantitative operation of *coordination*. Instead Rick was persistent in constructing rates through the *comparison* operation.

Rick is not an anomaly in that university students that assimilate with fewer than three levels of units exist in introductory calculus courses and appear to be at a higher risk of not finding success in such courses (Boyce, Grabhorn, & Byerley, 2020; Byerley 2019). Specific to our report, Johnson (2015) conjectured that sole reliance on the *comparison* operation could explain students' struggles with rates. This report builds a connection between Johnson's (2015) work investigating the quantitative operations involved in constructing rates and our previous research investigating the constraints and affordances of students' units coordinating activity as they develop understandings of calculus concepts. Further, Johnson left as an open question how students develop the *comparison* and *coordination* operations. We conjecture that engaging in the *coordination* operation (constructing a rate so that at least one of the quantities involved is continuously changing) requires assimilating with three levels of units.

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