ACKNOWLEDGING NON-CIRCULAR QUANTIFICATIONS OF ANGULARITY

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In recent years, researchers have advocated for measuring angles by measuring circular arcs (i.e., circular quantifications of angularity). Leveraging results from a teaching experiment with ninth-grade students, I demonstrate the existence of non-circular quantifications of angularity, which have not previously been acknowledged in existing empirical research or standards.

Keywords: Cognition, Geometry and Geometrical and Spatial Thinking, Measurement

Angularity is an important geometric attribute throughout K–12+ curricula. It arises in many contexts including classifying shapes, congruence, similarity, transformations, construction, proof, coordinate systems, and trigonometry. Despite this prevalence, few studies have investigated how students reason about angularity (Smith & Barrett, 2017). At the undergraduate level, researchers have argued that robust quantifications of angularity are critical for trigonometry (Akkoc, 2008; Moore, 2013). However, research on angularity with high school students is especially scarce. This presents a problem. In fact, Moore (2013) noted, "future studies that investigate secondary students' quantification of angle measure are needed…" (p. 243). To this end, I conducted a teaching experiment with ninth-grade students to understand how they quantified angularity. In this report, I elaborate the quantifications of angularity indicated by two students and consider implications of these results.

Theoretical Components and a Hypothesis

This study was informed by principles of quantitative reasoning (Thompson, 1994; 2011). A quantity is an individual's conception of a measurable attribute of an object or situation; quantities are mental constructions consisting of three interrelated components: (a) an object, (b) an attribute, and (c) a quantification. A quantification involves a collection of mental operations that an individual could carry out to measure an attribute or interpret a measurement value in a given context. For example, upon assimilating an angle model an individual might establish a goal of determining how open the angle model is in degrees; alternatively, an individual might be asked to consider how to make a one-degree angle. In these instances, the collection of mental operations activated would be components of the individual's quantification of angularity.

Following Thompson's (2008) first-order conceptual analysis, Moore (2013) elaborated that quantifying angularity involves (a) considering a circle centered at an angle's vertex, (b) making a multiplicative comparison of two lengths (e.g., arc length and circumference), and (c) recognizing this ratio is invariant across all possible circles centered at the angle's vertex; for example, a one-degree angle "subtends 1/360 of the circumference of any circle centered at the vertex of the angle" (p. 227). This approach is compatible with the CCSSM standards, where angle measure is explicitly introduced in Grade 4. I refer to these quantifications of angularity as *circular quantifications of angularity* because they leverage multiplicative comparisons of arcs and other circular lengths (e.g., circumference, radius, etc.). Circular quantifications of angularity yield coherent interpretations for angle measure across standard units of angular measure; thus, such quantifications of angularity are productive, particularly for the study of precalculus mathematics and beyond. However, circular quantifications of angularity are sophisticated, and angle measure is introduced relatively early in curricula. Therefore, it is reasonable to question: Is it possible for students to quantify angularity in

In: Sacristán, A.I., Cortés-Zavala, J.C. & Ruiz-Arias, P.M. (Eds.). (2020). *Mathematics Education Across Cultures: Proceedings of the 42nd Meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education*, Mexico. Cinvestav / AMIUTEM / PME-NA. https:/doi.org/10.51272/pmena.42.2020

other ways? Might these other quantifications support students in later constructing circular quantifications of angularity?

When one discusses the measure of an angle, one is describing the size of the interior of the angle (Hardison, 2019). The major hypothesis investigated in the present study was that students might establish productive non-circular quantifications of angularity by enacting extensive quantitative operations on angular interiors. Extensive quantitative operations are operations that introduce units (Steffe, 1991). In length and area contexts, Steffe & Olive (2010) provide numerous examples of such operations including iteration (imagining making and uniting copies of an established unit to produce a composite whole) and partitioning (imagining the simultaneous production of equal-sized parts within an established whole).

Methods

The data and analyses presented in subsequent sections are drawn from a teaching experiment (Steffe & Thompson, 2000; Steffe & Ulrich, 2013) conducted over an academic year in the southeastern U.S. with four ninth-grade students. At the time of the study, all students were enrolled in a first-year algebra course. The overarching goal of the teaching experiment was to investigate how the students quantified angularity and how these quantifications changed throughout the study (see Hardison, 2018); the author served as teacher-researcher for all teaching sessions. Throughout the study, students engaged in mathematical tasks involving rotational angle models (e.g., rotating laser) and non-rotational angle models (e.g., hinged wooden chopsticks). Each student participated in 13–15 video-recorded sessions, which were conducted individually or in pairs approximately once per week outside of their regular classroom instruction; each session was approximately 30 minutes in length. The records of students' observable behaviors (e.g., talk, gestures, written responses, etc.) were analyzed in detail via conceptual analysis (Thompson, 2008; von Glasersfeld, 1995). In this report, the activities of two students, Bertin and Kacie, are foregrounded to illustrate the existence of non-circular quantifications of angularity and to evidence that the construction of circular quantifications of angularity can be supported by non-circular quantifications.

Data and Findings

The results in the following sections are structured around the analysis of four purposefully selected examples of mathematical interactions with Bertin or Kacie.

Angular Repetition and Iteration

To establish models for students' ways of reasoning at the onset of the teaching experiment, students were asked to solve a variety of tasks. One such task involved two pairs of hinged wooden chopsticks: one short pair that could be freely adjusted and one long pair which was fixed. Each student was asked to set the short pair of chopsticks to be four times as open as the long pair of chopsticks. When presented with this task, Bertin proceeded by immediately tracing four adjacent copies of the long chopsticks on a piece of paper and setting the short pair of chopsticks to contain these four adjacent copies.

I refer to Bertin's physical actions as angular repetition. Through angular repetition, Bertin produced an angle model four times as open as the given angle model. Because Bertin engaged in angular repetition without hesitation, I infer he imagined uniting adjacent copies in visualized imagination prior to his physical actions. In other words, the immediacy of Bertin's activities suggests an anticipation indicative of the mental operation of angular iteration. Nothing in Bertin's observable activities indicated that Bertin was considering circular arcs as he solved this task; instead, the figurative material subjected to angular iteration was the interior of the given angle model. Because this task was from his initial interview, Bertin's way of reasoning was previously established and not engendered by the teacher-researcher. Thus, Bertin's activities indicated he may have constructed a non-circular quantification of angularity prior to the study.

One-Degree Angles

Three months later, Bertin was asked how to make an angle with a measure of one degree. Bertin replied, "If you get a ninety-degree angle [*gestures a right angle*], you can divide that into nine so it would be like ten degrees each, and then you can divide each one of those into ten, but it would need something like really really small to write with." The gesture Bertin enacted indicated he first brought forth a familiar angular template in visualized imagination, specifically a right angle. His response also indicated he had assigned this right-angle template a measure of 90°, thereby positing it as a composite unit. Bertin then imagined partitioning the right angle into nine 10° parts, each of which he subsequently partitioned into ten 1° subparts. Thus, Bertin indicated producing 90 one-degree angles within a right angle in visualized imagination. As in the previous example, Bertin's way of reasoning did not leverage circles or arcs; instead, Bertin demonstrated he had established a normative conception of a one-degree angle via extensive quantitative operations enacted on the interior of a familiar angular template.

Contraindication of a Circular Quantification of Angularity

Five months into the study, Bertin was presented with tasks involving central angles to determine whether he had constructed a circular quantification of angularity. In one task, he was asked to determine the measure of a central angle in degrees, given that the length of the green subtended arc was one inch and the green circle's circumference was six inches (Figure 1 left).

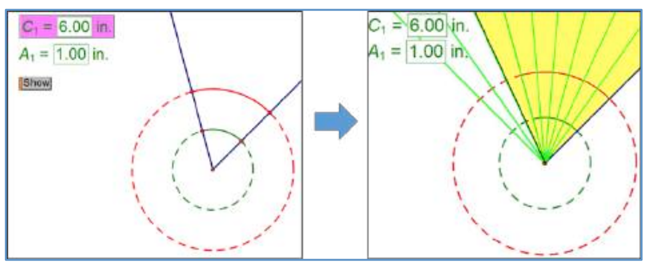

Figure 1: A Central Angle Task (left) and a Model of Bertin's Solution (right)

After an approximately 10-second pause, Bertin tentatively responded, "like seventy," and explained that he "kind of based off of ninety degrees" as he dragged the cursor to form a right angle containing the given central angle. When pressed for how he might precisely determine the measure, Bertin indicated with the cursor that he imagined partitioning the right angle into ten-degree parts; he then counted how many of these parts were contained in the central angle's interior. The green lines in Figure 1 (right) approximate how Bertin dragged the cursor to indicate ten-degree parts. Afterwards, Bertin reiterated, "it's like around seventy somewhere."

Bertin's activities indicated his reliance upon a non-circular quantification of angularity and were remarkably similar to his production of one-degree angles: he started with a familiar template (a right angle); posited this right-angle template as a 90° composite unit; and he partitioned it into nine 10° parts, which he leveraged to solve the task at hand. Bertin's solution is commendable; however, notably absent are any reference to the given measures for the arc or circumference. Thus, Bertin's activities contraindicate a circular quantification of angularity.

Evidence That Non-Circular Quantifications Can Support the Circular Counterpart

To illustrate that non-circular quantifications of angularity might support the construction of the circular counterpart, I present and analyze Kacie's activities on a final interview task involving a central angle. At this point, Kacie had developed a non-circular quantification of angularity similar to

Bertin's. In particular, Kacie had established the following way of reasoning: if n adjacent copies of an angular section exhaust a full angle, then the angle has a measure of $360° \div n$. The central angle task from Kacie's final interview involved determining the measure of a blue central angle subtending a green arc 3.47 cm long in a circle of circumference 22.83 cm. Kacie's reasoning is described in the transcript, which has been edited for brevity.

T: How would you determine the measure of the blue angle?
K: Um, [*11s pause*]. You could subtract, um, the three point four seven and the twenty two point eighty three. And that might give you your measurement. Because that's what that angle is like that – well, no. Just kidding. [*16s pause*]. Yeah. I guess you could subtract.
T: And what would that subtraction tell you?
K: Um [*4s pause*]. No! Wait. You could do twenty two point eight three divided by – wait, no. Yeah. Divided by three point four seven and that would give you the number of times the angle would go around the circle. And then you could do … three hundred sixty divided by that number and then that would give you the measurement of the angle.
T: Can explain why that works?
K: … well twenty two point eighty three divided by three point forty seven … would give you a number of how many times the green arc could go around the circle. And then that would give you how many times the blue angle would need to go to the circle to reach back to its starting point. And then if you did three hundred and sixty divided by the number of times the blue angle needed to go around it would give you the measurement

Kacie used the known arc length as a unit for measuring the known circumference. She considered the quotient of these lengths (i.e., 22.83÷3.47) without enacting the numerical division and interpreted this quotient as how many times the green arc "could go around the circle." Kacie also interpreted this quotient in terms of the central angle, which indicated she mentally united the arc and the central angle and was subjecting these united objects to the same mental operations. Having established the number of adjacent copies of the central angle needed to exhaust a full angle, Kacie relied on a previously established way of reasoning to solve the task. In short, Kacie was able to solve this task involving arc length by leveraging the non-circular quantification of angularity she had previously established.

Discussion, Conclusions, and Implications

Bertin and Kacie developed powerful non-circular quantifications of angularity reliant upon (a) establishing mental templates for familiar angles, (b) positing these familiar templates as composite angular units, and (c) making and measuring other angles via the application of extensive quantitative operations to angular interiors. These non-circular quantifications of angularity have not previously been identified and celebrated in empirical literature. Such quantifications are productive and should be recognized in classrooms and curricular standards along with circular quantifications. I hypothesize non-circular quantifications of angularity naturally precede, and are necessary for constructing, circular quantifications of angularity. Future studies are needed to investigate this hypothesis; however, Bertin's spontaneous angular repetition during the initial interview and his description of one-degree angles evidence that non-circular quantifications can precede circular quantifications, and Kacie's activities evidence that non-circular quantifications of angularity can support the circular counterpart. Additional research is needed to determine the prevalence of circular and non-circular quantifications of angularity at various grade levels.

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