# "SOLVING VERSUS RELATING": PRE-SERVICE TEACHERS' CONFLICTING IMAGES OF FORMULAS AND DYNAMIC CONTEXTS

Irma E. Stevens
University of Michigan
istevens@umich.edu

*Researchers have identified both the affordances of engaging students in symbolization activities and students' difficulties in meaningfully representing contexts through algebraic expressions/formulas. In a semester-long teaching experiment, two pre-service teachers demonstrated their conflicting meanings for formulas with their images of a context when engaging in a task about a dynamic geometric object. The two students could construct both normative formulas by reasoning with a context and descriptions of covariational relationships between quantities within the context, but both still struggled to relate their formulas and quantitative relationships to one another. This result highlights the importance of attending to what students' formulas mean to them, which for the students in this study, could be either a way of "solving" or "relating" quantities.*

Keywords: Precalculus, Algebra and Algebraic Thinking, Cognition, Teacher Education - Preservice

Several researchers have identified students' difficulties with symbolization within formulas, equations, etc. and others have illustrated students' ability to construct their own representational systems (e.g., Izsák, 2003). To support pre-service teachers in working with their future students, it is important to start with understanding what they know and similarly, understanding where perturbations (i.e., cognitive conflict) might occur. In this study, I explore two secondary mathematics pre-service teachers' (heretofore, students') meanings for formulas, particularly focusing on the relationship between students' images of context (i.e., the quantities they construct within contexts) and their associated formulas. To do so, I draw on two main bodies of research: symbolization activity and covariational reasoning—reasoning involving how two quantities change in tandem with each other (Carlson, Jacobs, Coe, Larsen, & Hsu, 2002). The research objective of this study builds on these researchers' findings through a teaching experiment with two students and is focused on learning about the mental operations involved in students' construction of formulas via reasoning with dynamic objects. Specifically, here, I focus on the students' meanings for an area formula for a parallelogram. The two students expressed conflicting meanings for a formula and its associated context. I describe these two students' meanings for their formulas and discuss the implications on students' symbolization activity based on their conflicting meanings and how they resolved them.

## Background and Theoretical Perspective

In an effort to distinguish between terms used throughout the results section, I adopt Thompson and Carlson's (2017) definitions of constants and variables as students envisioning the following: a *constant* is an image involving a quantity as having a value that does not vary ever and a *variable* involves a quantity's value varying within a setting. I further define an *undetermined constant* as one in which the individual considering a quantity has not established a unit of measure, but anticipates needing to do so in order to produce a value (cf., *unknown* constant). The definition of variable relates to the notion of covariational reasoning proposed (i.e., the quantities co-varied are variables) and is also compatible with Küchemann's (1981) definition of a variable as a letter "seen as representing a range of unspecified values, and a systematic relationship is seen to exist between two such sets of values" (p. 104). Lastly, I emphasize that an individual using a letter (or any other marking) as a symbol for a constant, parameter, or variable requires the individual to re-present that

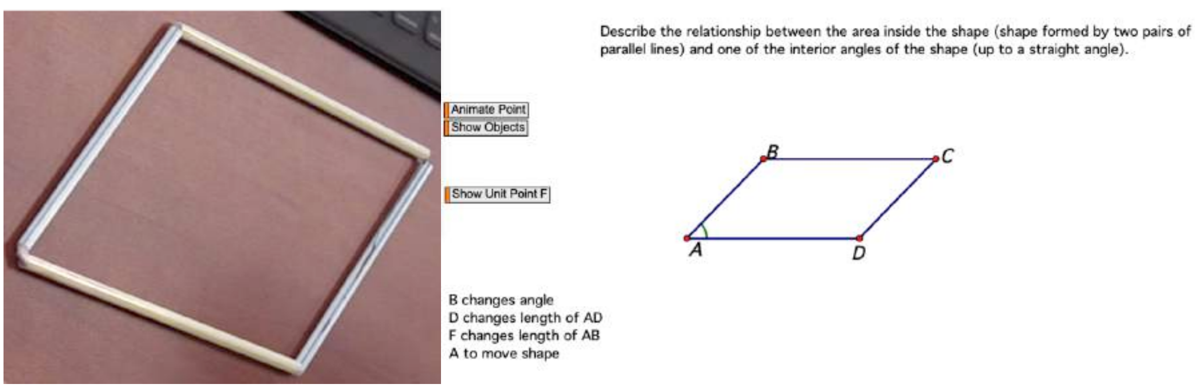
In: Sacristán, A.I., Cortés-Zavala, J.C. & Ruiz-Arias, P.M. (Eds.). (2020). *Mathematics Education Across Cultures: Proceedings of the 42nd Meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education*, Mexico. Cinvestav / AMIUTEM / PME-NA. https:/doi.org/10.51272/pmena.42.2020

letter as a quantity or quantitative relationship within a situation. That is, a letter in itself is not a representation; the marking is the figurative material that results from an individual's operations.

## Methods and Task

This study was part of a semester-long teaching experiment (Steffe & Thompson, 2000) with two secondary mathematics pre-service teachers at a large public university in the southeastern U.S. The students were selected from a pre-calculus secondary content course based on their results of a modified version of the *MMTSM* assessment (Thompson, 2012) and a pre-interview showing that the students had differing ways of reasoning about quantitative relationships. This report focuses on a task that occurred in Lily's hour-long teaching sessions 9-12 (of 12) and Dahlia's teaching sessions 6-9 (of 10). As a result of open and axial coding of the video recordings and transcripts of the lessons and applying the definitions for constants and variables, I describe how Lily and Dahlia constructed and interpreted their conflicting images.

I use the *Moving Angles Task* to discuss students' representational activity and construction of conflicting images. In this task, students were given the manipulative in Figure 1a and the prompt, "Describe the relationship between the area inside the shape (shape formed by two pairs of parallel lines) and one of the interior angles of the shape (up to a straight angle)." After initial discussions, both students received a sketch in a dynamic geometric environment (DGE) (Figure 1b) to support their exploration (which also included dynamic magnitude bars which are outside of the scope of this paper). The data was analyzed using generative and axial approaches (Corbin & Strauss, 2008) in order to construct models of the *mathematics of the students* (Steffe & Thompson, 2000). For a more detailed description of this task and insights into how this task has supported PSTs' covariational reasoning with equations, see Stevens (2018).



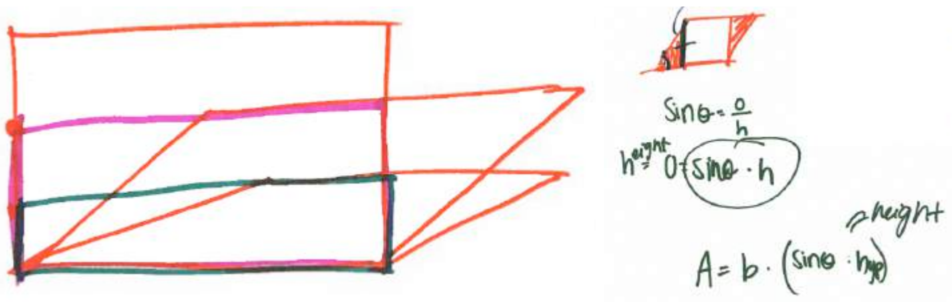

**(a)** **(b)**

**Figure 1: The Moving Angles Task (a) manipulative and (b) sketch within a DGE**

## Results

### Lily's Conflicting Images

Lily originally focused on exploring the covariational relationships between angle measure (specifically for $\angle DAB$), height of the parallelogram, and the area of the parallelogram. After much deliberation, she concluded (as illustrated in Figure 2a) that equal changes in height corresponded with equal changes in area (i.e., "when the height was partitioned in decreasing equally, so was the area"), and in turning the angle clockwise from a right angle, the angle "decreases by decreasing amounts."




**(a)** **(b)**

**Figure 2: (a) Lily’s exploration of covariational relationships within a context (b) Lily’s reasoning with a static parallelogram**

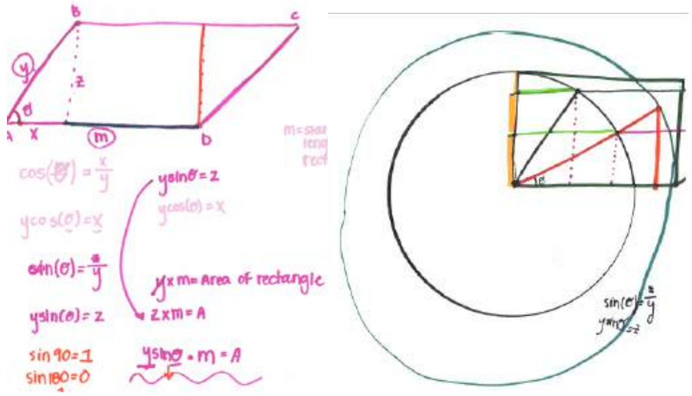
After her exploration, I asked Lily to “write an equation that represents the relationship that you’re talking about between the angle measure and the area of the parallelogram.” After several minutes, Lily drew a new parallelogram and stated that, “You can find this height [*of the parallelogram*] using sine and cosine” and produced the normative formula $A=b(\sin(\theta)*hyp)$, where $A$=area of parallelogram, $\theta$= m∠*DAB* in Figure 1b, and *hyp*= length of the hypotenuse in triangle in Figure 2b. In contrast to her drawing in Figure 2a, Lily said she did not see angle measure and height changing in her drawn parallelogram in Figure 2b. I interpret this description to indicate that Lily re-presented the symbols in her formula as undetermined constants of quantities she constructed from a static shape.

Lily indicated that her formula conflicted with her image of the relationships between quantities in the dynamic context, “Because I don’t know how to talk about it when I know this is true [*pointing to statement that from 0 to 90 degrees, angle measure increases so height increases so area increases*]. I don’t know how to relate it [*her statement*] to this part [*her parenthetical in her formula*].” She stated that the confusion stemmed from her understanding of her formula (Figure 2b) as “just solving, not relating,” where relating referred to seeing quantities (i.e., angle measure, height) as changing. In sum, to her, Lily, in constructing her formula, thought she was appropriately re-presenting a procedure for calculating area measures for static parallelograms but not the covariational relationships she constructed through her reasoning with the dynamic parallelogram.

## Dahlia’s Conflicting Images

Like Lily, Dahlia identified a non-linear relationship between angle measure and area of the given shape, and she constructed a formula similar to Lily’s formula in Figure 2b using similar reasoning (Figure 3a). Unlike Lily, Dahlia also provided a unit circle meaning for sine and re-presented the segment corresponding to the hypotenuse of the right triangle in Figure 3a also as both a hypotenuse of a triangle and the radius of the circle (see Figure 3b). Moreover, she re-presented a relationship between *changing* quantities within her drawn parallelogram and formula; she described $y$ as “not moving” and $z$ and $\theta$ as “changing” in her figure and formula.

Nevertheless, Dahlia could not answer the question, “Why would we multiply a portion of the radius [*her description of* sin*(θ)*] by the radius [*her description of y*]?”. Thus, although Dahlia could construct a formula that re-presented varying quantities in a situation by reasoning with trigonometric relationships, her formula still conflicted with her image of the context. This conflict occurred because she thought that to calculate the area of a parallelogram, she would need to multiply the base length and height of the parallelogram together, but her formula indicated to her that the side length, $h$, of the parallelogram was also needed to obtain an area measurement. Thus, although Dahlia thought she appropriately re-presented her image of dynamic quantities in the context as a formula based on her reasoning with trigonometric ratios, she struggled to relate the symbols to her image of the context.



**(a) (b)**

**Figure 3: (a) Dahlia’s formula for the area of a parallelogram with cosine related work dimmed for the reader (NOTE: m refers to AD, not the underlined segment in a different color) and (b) Dahlia’s diagram showing a unit circle approach for the sine relationship**

## Discussion and Implications

Researchers have often praised the symbolization activity of students developing their own symbols through representational activity, and this study is not an effort to discourage the use of contexts to support students’ meanings for formulas, equations, symbols, etc. Rather, this study indicates the importance of attending to the ways in which students’ symbolization activity re-presents their images of quantities and their relationships within given contexts. More specifically, Lily’s example indicates the importance of attending to students’ meanings for formulas (equations, functions, etc.) as ways to “solve for” or “figure out” values for constant quantities within static situations. This way of thinking about formulas was problematic for Lily even when she produced a normative formula because, for her, she was not re-presenting relationships between changing quantities with this formula. More generally, this view of formulas is problematic in students’ construction of variables because variables occur when a student re-presents values varying within a (dynamic) setting. Lastly, Dahlia’s example points to the importance of understanding students’ construction and role of units within their symbolization activity, particularly in regards to measurement contexts. By perturbing these meanings for formulas by attending to, for example, the role of units or the idea of a symbol as representative of a variable, students can accommodate their meanings for formulas to fit with their images of quantitative relationships in the context.

## Acknowledgments

This paper is based upon work supported by the NSF under Grant No. (DRL-1350342). Any opinions, findings, and conclusions or recommendations expressed in this material are those of the authors and do not necessarily reflect the views of the NSF.

## References

Carlson, M. P., Jacobs, Coe, Larsen, & Hsu (2002). "Applying covariational reasoning while modeling dynamic events: A framework and a study." *Journal for Research in Mathematics Education* 33(5): 352–378.

Corbin, J. M., & Strauss, A. (2008). *Basics of Qualitative Research: Techniques and Procedures for Developing Grounded Theory* (3rd ed.). Thousand Oaks, CA: Sage.

Izsák, A. (2003). ""We want a statement that Is always true": Criteria for good algebraic representations and the development of modeling knowledge." *Journal for Research in Mathematics Education* 34(3): 191-227.

Küchemann, D. (1981). Algebra. *Children's Understanding of Mathematics:11-16*. J. Murray. London, Alden Press: 102–119.

Steffe, L. P. and P. W. Thompson (2000). Teaching experiment methodology: Underlying principles and essential elements. *Handbook of research design in mathematics and science education*. R. A. Lesh and A. E. Kelly. Hillside, NJ, Erlbaum: 267-307.

Stevens, I. E. (2018). The parallelogram problem: Supporting covariational reasoning in the construction of formulas. In Hodges, T. E., Roy, G. J., & Tyminski, A. M. (Eds.)*, Proceedings of the 40th annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education* (pp. 624–627). Greenville, SC: University of South Carolina & Clemson University.

Thompson, P. W. (2012). Project ASPIRE: Defining and Assessing Mathematical Knowledge (Meanings) for Teaching Secondary Mathematics. Retrieved from http://aspire.mspnet.org/index.cfm/24324

Thompson, P. W. and M. P. Carlson (2017). Variation, covariation, and functions: Foundational ways of thinking mathematically. *Compendium for research in mathematics education*. J. Cai. Reston, VA, National Council of Teachers of Mathematics: 421–456.