# CHILDREN'S INTEGER DIVISION: EXTENDING ANALOGIES AND DIRECT MODELING

## DIVISIÓN INTEGRAL DE NIÑOS: EXTENDIENDO ANALOGÍAS Y MODELADO DIRECTO

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Even famous mathematicians, such as Pascal and Diophantus, rejected the possibility of negative integer solutions to algebraic equations (Bishop et al., 2011, 2014; Dehaene, 1997; Gallardo, 2002). Historical struggles make sense when we look at the limitations of physical models and instructional models for integers (Martínez, 2006; Peled & Carraher, 2008; Schwarz et al., 1993–1994; Wessman-Enzinger, 2019). Consider modelling 2 – -5 with discrete chips. Although this is possible with two-colored chips or tiles (Flores, 2008) and using concepts of zero pairs (e.g., Dickman & Bofferding, 2017), it is not necessarily intuitive and has constraints (Murray, 2018; Vig et al., 2014). These challenges in the physical embodiment of negatives integers extend to thinking and learning about multiplicative situations (e.g., -4 × -5). It is important to understand the ways that children use and extend their previous whole-number to negative integers in order to best support their thinking in our classrooms.

## Theoretical Framing: Division with Integers

Cognitively guided instruction illustrates problem types and strategies for thinking about division (Carpenter et al., 2015). We can describe *partitive division* as having a total amount, dividing that amount by a certain number of groups, and determining the amount per group (total ÷ number of groups = amount per group). We can describe *measurement division* as having a total amount, dividing that total by the amount per group, and determining the number of groups (total ÷ amount per group = number of groups). These two-problem types, partitive and measurement division, are well-accepted in the field (e.g., Jansen & Hohensee, 2016; Nueman, 1999). Children's *direct modeling strategies,* as they solve division problems, often align with these problem types (e.g., Carpenter et al., 2015; Mulligan & Mitchelmore, 1997). Despite constraints with the physical embodiment of the negative integers (Martínez, 2006), direct modeling can be productive for students to extend their whole number reasoning to multiplication with negatives (Carpenter & Wessman-Enzinger, 2018).

When children engage in solving addition and subtraction problems with negative integers, they construct a variety of productive strategies, including drawing on analogies (Bishop et al., 2014, 2016, 2018; Bofferding & Wessman-Enzinger, 2018; Wessman-Enzinger, 2019; Whitacre et al., 2017). For example, they productively compare -2 + -3 to 2 + 3 with analogical reasoning (Bishop et al., 2016, 2018; Bofferding, 2010, 2011; Whitacre et al., 2017). Yet, sometimes these analogies break down for multiplication (Carpenter & Wessman-Enzinger, 2018)—a student incorrectly determined $-4 \times -2$ is -8 with a logical analogy to $4 \times 2 = 8$.

The work reported in this research brief extends the discussion on children's thinking about negative integers by highlighting the ways two Grade 5 children reasoned with analogies and direct modeling with integer division.

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## Methods

Alice and Kim, from a rural Midwest school in the United States, participated in a 12-week teaching experiment (Steffe & Thompson, 2000) centered on integer addition and subtraction. The children in this study became accustomed to exploring mathematical ideas, struggling, and sharing their invented thinking. The session described here departed from the study on addition and subtraction and is an exploratory session on division. I wondered how children would *construct* integer division. In this session, two of the fifth-graders, Alice and Kim, worked together on three different division open number sentences with negative integers for fifty minutes. The three division problems they discussed in this session included: -24 ÷ ☐ = -2, ☐ ÷ -3 = -10, and -21 ÷ ☐ = 7. Teaching experiment methodology draws on using conceptual analysis, which requires examining the thinking of individuals iteratively (Thompson, 2008).

## Results and Discussion

Alice and Kim used analogical thinking and direct modeling strategies that drew mostly on partitive thinking as they solved division problems with negative integers. Their ways of thinking, as well as affordances and constraints, are addressed in this section.

### Analogical Thinking with Integer Division

Alice and Kim solved 24 ÷ ☐ = -2 first in the session. Both Alice and Kim's inaugural approach of this open number sentence included constructing analogies to 24 ÷ 12 = 2 and reasoning that -24 ÷ -12 = -2. Alice shared, "I think it's -12 because twenty-four divide by twelve is two. And, these two are negative (pointing at -24 and -2), so I thought that this would have to be negative." And, Kim shared, "Yeah. There's both negatives (shrugs). I just compared it to something easier. I just did 24 divide by twelve."

For ☐ ÷ -3 = -10, Alice and Kim made analogies again. They compared ☐ ÷ -3 = -10 to 30 ÷ 3 = 10. Kim reasoned, "You can multiply these two (points at -10 and -3 on the paper), but in a negative way, in the negative side. So, three times ten is thirty, so just add the negative symbol onto it." Alice and Kim both reasoned that -30 is the solution to ☐ ÷ -3 = -10 to because 30 ÷ 3 = 10.

There are similarities in analogical reasoning across these integer division number sentences; Alice and Kim compare both to fully positive integer number sentences. Children often draw on analogies productively for both addition and subtraction with negative integers. In fact, problem types like -2 + -5 and -6 – -3 are readily solved by children who see negative integers for the first time by comparing -2 + -5 to 2 + 5 and -6 – -3 to 6 – 3 (e.g., Bofferding, 2011; Whitacre et al., 2017). Note the structural similarities between these analogies: addition and subtraction with negatives are compared to fully positive numbers sentences, but result in correct solutions. Therefore, it is not surprising that Alice and Kim applied an analogy that worked well for integer addition and subtraction to these open number sentences with division, -24 ÷ ☐ = -2 and ☐ ÷ -3 = -10. However, this structurally similar analogy (comparing 24 ÷ 12 = 2 to -24 ÷ ☐ = -2) does not result in the correct solution.

Although the analogies described above did not support correct solutions, Alice solved -21 ÷ ☐ = 7 correctly using an analogy. Alice thought the open number sentence -21 ÷ ☐ = 7 was challenging division open number sentence type; Alice wrote "nope" and the solution -3 (-21 ÷ -3 = 7) and Kim wrote "wung it!" and the solution 3. Kim reasoned that the solution to -21 ÷ ☐ = 7 is 3. She made an analogy similar to what she did previously, comparing -21 ÷ 3 to 21 ÷ 3. Kim noticed something different in the structure, as she wrote "wung it" and shared uncertainty.

Alice suggested the correct solution of -3 when solving -21 ÷ ☐ = 7. Alice was unsure of her solution and verbalized her struggle. She, for example, wrote "nope" on her paper conjecturing, "it's wrong." She shared the challenge with the structure of this integer division problem: "I mean it doesn't make sense that a negative divided by a negative would equal a positive." Alice compared -

21 ÷ ☐ = 7 to -21 ÷ 3 = -7 and 21 ÷ 3 = 7, conjecturing -21 ÷ -3 = 7. Although unsure about -3 a solution, she maintained a strong confidence the answer is not 3, stating how numbers like -21 and 21 are different.

One challenge analogical reasoning that analogies that work well for integer addition and subtraction (comparing -2 + -3 to 2 + 3), to not readily extend to integer division (comparing -24 ÷ -12 = -2 to 24 ÷ 12 = 2). An affordance of analogical reasoning is that Alice was able to comparing -21 ÷ ☐ = 7 to 21 ÷ 3 = 7 and know that solution could be negative.

**Direct Modeling: Partitive and Measurement Thinking with Integer Division.** Alice used a direct modelling strategy for -24 ÷ ☐ = -2 when prompted to explain her analogical reasoning above. Both Kim and Alice used direct modeling strategies for justifying their analogies as they solved ☐ ÷ -3 = -10. For the last open number sentence in the session, -21 ÷ ☐ = 7, both Kim and Alice used analogical reasoning only.

Alice drew out tallies and groups, a component of direct modeling, because she wanted to justify the analogy she made when first solving -24 ÷ ☐ = -2. The transcript below illustrates the Alice's discussion about her tallies and group in Figure 1:

> I did two rows (pointing at tallies). They both had… they are negatives. And, there's two (draws two circles around groups of tallies). … Because if it was positive twelve, then this would be (points at the -2) … I don't know. … I did twenty-four (shows drawing with 24 tallies) and what I did is twelve and twelve and it makes two (points at two groups of 12 tallies in the drawing).

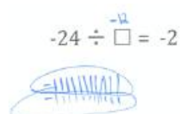
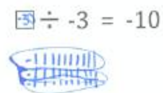



**Figure 1: Alice's use of direct modeling when solving -24 ÷ ☐ = -2**

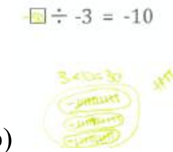
Alice created a total of 24 tallies, stating "they are negatives." As she made these 24 tallies, she placed them in two groups—she drew on a partitive division reasoning. Each of her groups has 12 tallies (see Figure 1), which represents -12. Alice actually illustrated -24 ÷ -12 = +2 with her direct modeling, and even refers to "two groups" (+2) instead of -2. She used partitive division because she doles out the tallies into two groups using how many tallies are in each group (12 tallies, or -12) as the unknown. For Alice, the number of groups was known, using +2 instead of -2, from the open number sentence. She stated that the amount in each group was unknown in -24 ÷ ☐ = -2, putting tallies in two groups and counting the tallies after.

One challenge that Alice faced is that her drawing is well-suited for -24 ÷ -12 = +2, instead of -24 ÷ 12 = -2. A second challenge that Alice encountered is making sense of what -2 groups with partitive division entails; indeed, -2 groups used in this way has physical has constraints. The physical constraints could potentially be why Alice states +2 groups instead of -2 groups. Suppose Alice used partitive reasoning and direct modeling in a way that resulted in the correct solution for -24 ÷ ☐ = -2. How could Alice physically take the existing 24 tallies that represent -24 and put the +12 tallies into -2 groups? Having -2 groups seems a bit ridiculous. However, in this case Alice could have used a direct modeling strategy with measurement division and it could have worked. For example, if Alice, instead, took her -24 tallies and looked for an unknown number of groups (12) that have -2 each in them, this could have worked. Using Alice's strategy with tallies, she could have started making groups of negative two or two tallies. If she did this, there would be 12 groups.

Both Alice and Kim drew on direct modeling strategies when justifying their analogies as they solved ☐ ÷ -3 = -10 (see Figure 2a and 2b, respectively). Both of their drawings produced for this open number sentenced used a direct modelling strategy similar to what Alice created (see Figure 1).



(a)



(b)

**Figure 2: Alice and Kim's use of direct modeling when solving ☐ ÷ -3 = -10**

The following excerpt of transcript highlights how Kim discussed her drawing in Figure 2b:

> Um, I did three groups. I did three groups. The negatives is, is the negatives symbol which it tells us that the answer is going to be a negative. These numbers are negatives. I did three groups of ten tally marks in each. Like how you do tally marks regularly, like three, four, and equal five. I did that. I screwed up, but it doesn't matter though. …These circles represent three groups. … Negative thirty is all of these combined together (motions around drawing). … Um, once you add all these up it equals negative thirty. And, you divide by negative three. …If you add all these up, it equals thirty, you can just divide it by three because of the three groups….The negative ten is in each of these (points at the tallies in the groups). I screwed up there's six in each of these, but there's supposed to be five.

Alice and Kim both used partitive reasoning for division, where they interpreted dividing by -3 as indication that there are +3 groups (not -3). They put ten tallies in each group, which represents -10 (albeit Kim has groups of 6 tallies instead of 5, but she counts them as five and recognizes the extra tallies). Both Alice and Kim determine that the solution is -30 to ☐ ÷ -3 = -10 by counting the total amount of "negative" tallies they have.

Kim interpreted the "÷ -3" in ☐ ÷ -3 = -10 as the number of groups positive three groups (+3), which is a challenge whether you use partitive division or measurement division. Number sentences like -24 ÷ ☐ = -2, can be supported by the students' partitive reasoning if they twelve groups of -2. But, is partitive or measurement division intuitive with ☐ ÷ -3 = -10? The open number sentence ☐ ÷ -3 = -10 differs from -24 ÷ ☐ = -2 in the sense that the positive number (i.e., ☐ or 30) is the dividend. Analogical reasoning may be more productive for ☐ ÷ -3 = -10.

## Concluding Remarks

This qualitative description of thinking about integer division highlights affordances and constraints of both analogies and direct modeling with integer division. Analogies for whole number addition and subtraction do not extend readily, but can be used productively. Although direct modeling had limitations with partitive division for -24 ÷ ☐ = -2, Alice's direct modeling highlighted the potential of measurement thinking with this particular open number sentence. Yet, both partitive and measurement definitions of division have limitations for number sentences like ☐ ÷ -3 = -10—highlighting potential for analogical reasoning.

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