STUDENTS' UNDERSTANDINGS OF THE TRANSFORMATIONS OF FUNCTIONS

Camille S. Burnett
Prairie View A&M University
csburnett@pvamu.edu

The goal of this study is to describe students' understandings of the transformations of functions in different representations based on an analysis of pilot interviews with two ninth and two twelfth grade students from the same urban public high school in Massachusetts, which serves a diverse community. Interview responses indicated that the students were unable to identify explicitly the type of transformation that described the relationship between two functions. Analyses of the interviews revealed that a student's flexibility in the use of representations of and approaches towards functions is an indicator of their understanding of functions and, therefore, that the ninth grade students interviewed have a less sophisticated understanding of functions than do the twelfth grade students interviewed.

Keywords: Algebra and Algebraic Thinking

Purpose of Study

Functions, as proposed by Schwartz and Yerushalmy (1992; Doorman & Drijvers, 2011; Schwartz, 1999; Zandieh et al., 2017), can be viewed as one of the fundamental objects of mathematics, and appears at all levels of the mathematics curriculum ranging from patterns in elementary school to real analysis in college mathematics. Landmark studies about the concept of function include, but are not limited to: (1) theoretical models on the development of the function concept; (2) teaching experiments that apply general theories to the specific concept of function; (3) students' and teachers' conceptions of functions; and (4) the use of technology in functions-based mathematics classes (e.g., Dubinsky & Harel, 1992b). The current study falls under the third category above, which includes more recent studies such as Ayalon and Wilkie (2019), Dubinsky & Wilson (2013), and Ronda (2015). It aims to contribute to this line of research by specifically analyzing and describing students' *understandings* of the *transformations* of functions in *different representations*. Thus far, researchers have identified that students experience difficulties with transformations of functions when asked to (a) visualize the transformations because of processing challenges with horizontal and vertical translations (Eisenberg & Dreyfus, 1994); (b) identify, graph and use transformations to solve problems because they have not interiorized the concept of function (Lage & Gaisman, 2006); and (c) translate functions because of cognitive and pedagogical obstacles (Zazkis et al., 2003). This study will describe some of the difficulties students encounter with transformations of functions, in particular, when asked to state the relationship, based on a transformation, between two given functions.

Theoretical Framework

Representations of Functions

There are various ways to represent a function. This study considers the algebraic, graphical, and tabular representations, which frequently occur in a high school math curriculum. The term algebraic representation refers to expressions or equations containing numbers and variables connected by mathematical operations. The term graphical representation refers to the Cartesian coordinate system, and the term tabular representation refers to a table of values displaying an input and an output.

In: Sacristán, A.I., Cortés-Zavala, J.C. & Ruiz-Arias, P.M. (Eds.). (2020). *Mathematics Education Across Cultures: Proceedings of the 42nd Meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education*, Mexico. Cinvestav / AMIUTEM / PME-NA. https:/doi.org/10.51272/pmena.42.2020

Conceptions of Functions

The most predominant distinction used for describing one's concept of a function is *process* versus *object*. A *process* conception of a mathematical concept is "a form of understanding of a concept that involves imagining a transformation of mental or physical objects that the subject perceives as relatively internal and totally under her or his control" (Dubinsky & Harel, 1992b, p. 20). An *object* conception of a mathematical concept is "a form of understanding of a concept that sees it as something to which actions and processes may be applied" (Dubinsky & Harel, 1992b, p. 19). One method for identifying one's conception of function is to consider one's approach towards functions, as is done in this study. One can have a pointwise or a global approach towards a function (Bell & Janvier, 1981; Janvier, 1978). For instance, if given the algebraic representation of a function and asked to create the graph, then a pointwise approach is to plot discrete points, and a global approach is to sketch the graph (Even, 1998).

Conceptions and Representations of Functions

A students' understanding of the concept of function can vary depending on the representation (Dubinsky & Harel, 1992a; Moschkovich et al., 1993). This is likely because the tabular representation is composed of discrete data points and requires a pointwise (process) approach, and the algebraic and graphical representations can be manipulated discretely (pointwise/process) or in their entirety (global/object). Further, to be able to translate between representations is associated with being able to transition between approaches (Even, 1998), and it is important to be able to move flexibly between representations and understandings (Moschkovich et al., 1993).

Methods

Participants

The participants included two high-performing ninth graders (Student 9-1 and 9-2) and two high-performing twelfth graders (Student 12-1 and 12-2), from the same public high school in an urban center in Massachusetts, which serves a diverse community. The ninth graders were learning basic algebra principles such as order of operations, as well as statistical concepts such as box plots; the twelfth graders were learning about polynomial functions and their characteristics. The students were selected by their mathematics teachers to participate in this pilot study based on their excellent grades and high skill level in their current mathematics class.

Individual Interviews

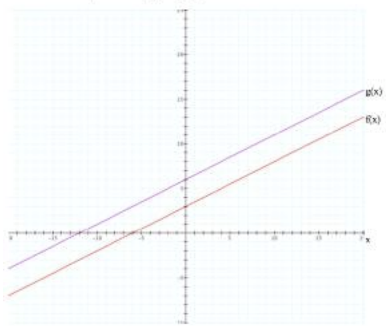
The individual interviews piloted seven questions pertaining to the definition of a function, and the transformations and comparisons of functions in different representations; however, this paper will focus only on the responses to the three interview questions regarding the transformations of functions (Questions 2, 3, and 4 – see Figure 1). In these questions, the participants were asked to state the relationship between the two given functions. The interviews were videotaped and lasted between 15 and 60 minutes, depending on the participant.

Analysis

The participants' responses to Questions 2, 3, and 4 were analyzed based on Figure 2, which was adapted from Moschkovich et al.'s (1993) study, and includes Even's (1998) approaches to functions – pointwise or global. If a participant's response was considered accurate, then it was coded with a 1, and if it was not, then it was coded with a 0. Also, the "most anticipated" response cells appear in boldface for each participant. The "most anticipated" responses were chosen based on two criteria: (1) it was in the representation in which the question was posed, and (2) it used an approach aligned with the representation. More specifically, for the second criterion, the graphical representation tends to evoke the object conception of function (Schwartz & Yerushalmy, 1992), while the tabular representation relies on discrete data points and, therefore, is more closely related to the process conception of function. Thus, each participant could have a "most anticipated" response, as well as accurate, non-anticipated responses.

Question 2

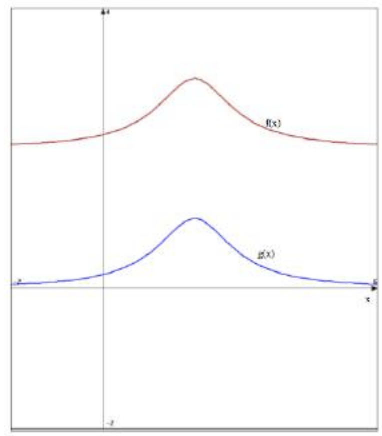
Given the following information for the functions *f(x)* and *g(x)*, what could be the relationship between *f(x)* and *g(x)*?



Note: Functions can be expressed in other representations, and relationship is a translation.

Question 3

Given the following information for the functions *f(x)* and *g(x)*, what could be the relationship between *f(x)* and *g(x)*?



Note: Functions cannot be expressed in other representations, and relationship is a translation.

Question 4

Given the following information for the functions *f(x)* and *g(x)*, what could be the relationship between *f(x)* and *g(x)*?

x	f(x)
-5	-8
-4	-5
-3	-2
-2	1
-1	4
0	7
1	10
2	13
3	16
4	19
5	22

x	g(x)
-5	-16
-4	-10
-3	-4
-2	2
-1	8
0	14
1	20
2	26
3	32
4	38
5	44

Note: Functions can be expressed in other representations, and the relationship is a dilation.

Figure 1: Interview Questions

	Representation		
Approach	*Tabular (T)*	*Graphical (G)*	*Algebraic (A)*
Pointwise			
Global			

Figure 2: Framework for Examining Representations and Approaches to Functions

Results

Response Samples

Question 2. Student 9-2 was unsure as to what the question was asking and kept referring to the graphs as being representative of information usually seen in a table or as an equation. The student then pointed out that the two lines are parallel, and deduced the slope of each function from their respective graphs, which was found to be the same. The student continued to examine the two functions and pointed out that they both have different x- and y-intercepts, and finally concluded, "They're both the same, I guess. They're both the same lines…just in different positions … One's higher, and one's lower. They're placed…they're the same…the same two lines…just placed different on the axis."

Question 3. Student 12-1 started by examining the graphs and stating that if the functions were expressed in tabular form, then the x-values would be the same for each function, but the output values would be different by a power or a multiple. The student showed great difficulty in explaining the relationship but was able to say, "The difference between the y-axis on either graph is going to be the number that you are going to be ... adding or multiplying to [g(x)]."

Question 4. Student 12-2 displayed high confidence in responding to the question and stated that "There's…you can find the difference between the two if you have this, and you say, …g(x)…, …x and y table is, um…in order to get that, all you have to do is multiply by two. Then, you would…then you could easily find g(x). And, you could plot it out. And, you could discover the, um…the slope…and the y-intercept. And, you could find out the equation."

Response Summary

The analysis of the participants' responses to Questions 2, 3, and 4 is summarized in Table 1.

Table 1: Results of Questions 2, 3, and 4

		Question 2			*Question 3*			*Question 4*		
		Representation								
Participant	*Approach*	*T*	*G*	*A*	*T*	*G*	*A*	*T*	*G*	*A*
Student 9-1	*Pointwise*	*0*	*1*	*0*	*0*	*1*	*0*	*1*	*0*	*0*
	Global	*0*	*1*	*0*	*0*	*1*	*0*	*0*	*0*	*1*
Student 9-2	*Pointwise*	*0*	*0*	*0*	*0*	*0*	*0*	*1*	*0*	*0*
	Global	*0*	*1*	*1*	*0*	*1*	*0*	*0*	*0*	*0*
Student 12-1	*Pointwise*	*0*	*0*	*1*	*0*	*1*	*0*	*1*	*0*	*0*
	Global	*0*	*0*	*1*	*0*	*1*	*0*	*0*	*0*	*0*
Student 12-2	*Pointwise*	*0*	*0*	*1*	*0*	*0*	*0*	*1*	*1*	*1*
	Global	*0*	*0*	*1*	*0*	*1*	*0*	*0*	*0*	*1*

Discussion

Interview responses indicated one significant observation in terms of students' understandings of *transformations*: the students were unable to identify explicitly the type of transformation that described the relationship between any of the pairs of functions but were able to use other descriptive words for the same. More specifically, none of the students used the terms translation, dilation, or transformation in response to any of the questions. Instead, they used words such as "higher" or "lower" on the graph to describe a translation, and "multiplied" to describe a dilation, which indicates a lack of mathematical vocabulary. Also, Student 12-1 was unable to specifically determine if the functions in Question 3 represented a translation or a dilation, which indicates a lack of understanding of transformations. These observations regarding difficulties with transformations need to be substantiated with more research.

Analysis of the interviews highlighted two significant findings in terms of students' *understandings* of functions in different *representations*: (1) students were able to be flexible between representations for functions, moving from one to another even though the question they were answering was presented in a single type of representation; and (2) students were able to be flexible between approaches towards and conceptions of functions, providing evidence that they could approach single questions and functions embedded in them in both a pointwise (process) and global (object) way. Given our assumption that moving across representations of functions and adopting both pointwise and global approaches towards functions provides evidence for students' flexibility and therefore greater sophistication in their understandings about functions ((Even 1998; Moschkovich et al. 1993), these findings lead us to the preliminary conclusion that the ninth grade students interviewed have a less sophisticated understanding of the concept of function than do the twelfth grade students interviewed because neither ninth grade student exhibited any flexibility between representations and only one exhibited flexibility between approaches. This preliminary conclusion needs to be substantiated with further research.

References

Ayalon, M., & Wilkie, K. J. (2019). Exploring secondary students' conceptualization of functions in three curriculum contexts. Journal of Mathematical Behavior, 56. https://doi-org.pvamu.idm.oclc.org/10.1016/j.jmathb.2019.100718

Bell, A., & Janvier, C. (1981). The interpretation of graphs representing situations. *For the Learning of Mathematics, 2*(1), 34-42.

Doorman M., & Drijvers P. (2011) Algebra in Function. In P. Drijvers (Ed.), Secondary Algebra Education (pp. 119-136). Sense Publishers.

Dubinsky, E., & Harel, G. (1992a). The nature of the process conception of function. In E. Dubinsky & G. Harel (Eds.), *The concept of function: Aspects of epistemology and pedagogy* (pp. 85-106). Mathematical Association of America.

Dubinsky, E., & Harel, G. (Eds.). (1992b). *The concept of function: Aspects of epistemology and pedagogy*. Mathematical Association of America.

Dubinsky, E., & Wilson, R. (2013). High school students' understanding of the function concept. *The Journal of Mathematical Behavior, 32*(1), 83-101. https://doi.org/10.1016/j.jmathb.2012.12.001

Eisenberg, T., & Dreyfus, T. (1994). On understanding how students learn to visualize function transformations. In E. Dubinsky, A. Schoenfeld, & J. Kaput (Eds.), *Research in collegiate mathematics education* (pp. 45-68). American Mathematical Society.

Even, R. (1998). Factors involved in linking representations of functions. *Journal of Mathematical Behavior, 17*(1), 105-121.

Janvier, C. (1978). *The interpretation of complex Cartesian graphs representing situations – studies and teaching experiments.* [Unpublished doctoral dissertation]. University of Nottingham, Shell Centre for Mathematical Education and Universite du Quebec a Montreal.

Lage, A. E., & Gaisman, M. T. (2006). An analysis of students' ideas about transformations of functions. *Proceedings of the 28th annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education. 2*, 23-30. Mérida, México: Universidad Pedagogica Nacional.

Moschkovich, J., Schoenfeld, A. H., & Arcavi, A. (1993). Aspects of understanding: On multiple perspectives and representations of linear relations and connections among them. In T. A. Romberg, E. Fennema, T. P. Carpenter (Eds.), *Integrating research on the graphical representation of functions* (pp. 69-100). Lawrence Erlbaum Associates, Inc.

Ronda, E. (2015). Growth points in linking representations of function: A research-based framework Springer Nature. *Educational Studies in Mathematics, 90*(3). 303-319. https//doi.org/10.1007/s10649-015-9631-1

Schwartz, J. L. (1999). Can technology help us make the mathematics curriculum intellectually stimulating and socially responsible? International Journal of Computers for Mathematical Learning, 4(2-3), 99-119.

Schwartz, J., & Yerushalmy, M. (1992). Getting students to function on and with algebra. In E. Dubinsky & G. Harel (Eds.), The concept of function: Aspects of epistemology and pedagogy (pp. 261-289). Mathematical Association of America.

Zandieh, M., Ellis, J., & Rasmussen, C. (2017). A characterization of a unified notion of mathematical function: The case of high school function and linear transformation. *Educational Studies in Mathematics 95*(1), 21-38. https//doi.org10.1007/s10649-016-9737-0

Zaskis, R., Liljedahl, P. & Gadowsky, K. (2003). Translation of a function: Coping with perceived inconsistency. *Proceedings of the 27th International Group for the Psychology of Mathematics Education Held Jointly with the 25th Psychology of Mathematics Education North America Conference,* 4, 459-468, Hawaii, USA.