FROM RECURSION TO INDUCTION: STUDENTS' GENERALIZATION PRACTICES THROUGH THE LENS OF COMBINATORIAL GAMES

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In this study we describe the generalization and justification practices of students in a highly selective summer mathematics program as they explore a sequence of problems from combinatorial game theory. We find that while study participants readily generate examples and reason recursively when analyzing Nim-like two-player combinatorial games and are able to reach valid conclusions about winning strategies in these games, they do not readily formalize their justifications into proofs using mathematical induction. We describe some obstacles that we observe in the transition between recursive reasoning and proof by induction.

Keywords: Advanced Mathematical Thinking, Problem Solving, Reasoning and Proof

Research in undergraduate mathematics education has identified obstacles to students' acquisition and understanding of mathematical induction as a proof strategy (*e.g.*, Brown, 2008; Dubinsky, 1989; Harel & Brown, 2008; Movshovitz-Hadar, 1993) and described the understandings that undergraduate students, including preservice teachers, have of induction (*e.g.*, Stylianides, Stylianides, & Philippou, 2007). Harel and Brown (2008) point out that in many standard instructional treatments of proof by mathematical induction (PMI), problems that exemplify the utility of the proof strategy can be categorized into *recursion* and *non-recursion* problems based on whether they involve recursive representations of functions or processes. The authors note that insistence on the use of PMI to solve non-recursion problems, in the absence of genuine intellectual necessity (Harel, 2013), can reinforce students' authoritative proof schemes.

One genre of problems that receives little attention in introduction-to-proof courses in the U.S. deals with *combinatorial games*, deterministic two-player games with perfect information (nothing concealed from either player). In a two-player combinatorial game with a finite set of possible game states that must terminate after a finite number of moves with one player winning, each game position can be characterized as affording a winning strategy to the next player to move (a *winning position*) or not (a *losing position*). The process of classifying positions in a game as winning or losing often involves *recursive reasoning* (Lannin, Barker, & Townsend, 2006), reasoning about cases of a problem by referencing previously established cases. This recursive reasoning can then be generalized to describe the set of all winning or losing positions for the game, and this reasoning can be distilled into a proof by mathematical induction that formally verifies and explains this description. We hypothesize that the genre of problems about combinatorial games offers a potential setting in which PMI can fulfill the explanation function of proof (De Villiers, 1990; Hanna, 2000). While problems involving combinatorial games have some complexity not associated with typical "textbook" problems, such as requiring complex induction hypotheses (incorporating assumptions about both winning and losing positions), we view them as possessing the exploratory nature and recursive structure needed to create intellectual necessity for inductive proof.

In analyzing students' work on problems that invite recursive reasoning about examples and eventually call for generalization of this reasoning, we categorize students' generalization activities into *result pattern generalization*, in which a general insight is obtained by observing regularity in results of calculations, or as *process pattern generalization*, in which this insight is backed by an understanding of regularity in the processes by which these results occur (Harel, 2001). We view

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process pattern generalization as a potential way of transitioning from recursive reasoning about specific cases of a problem to the development of a general inductive argument.

Guided by this framework, we address the following question: In what ways do students engage in recursive reasoning and generalization as they work on problems involving two-player combinatorial games, and to what extent do they formalize their justifications using PMI?

Method of Study

We report results of a case study (Yin, 2017) of students' collaborative work on a sequence of problems involving combinatorial games, with each case consisting of the mathematical discussion, arguments, and written and visual representations of a group of four students. The study took place in an informal summer mathematics program for students ages 13-18.

We selected sixteen students in the program who reported low levels of prior familiarity with puzzles about Nim-like games (based on an initial questionnaire), and assigned them to four groups of four students each. Each group then participated in a video-recorded task-based interview lasting approximately an hour and a half. Each group worked collaboratively on a sequence of five tasks, each of which asked participants to analyze a Nim-like two-player combinatorial game. The first three tasks are shown in Figure 1. At the conclusion of each problem, a researcher asked the students to summarize and justify their conclusions orally, and asked questions as needed to clarify our understanding of students' reasoning.

Problem 1
Player A and Player B are playing a game where there is one pile of 10 marbles. Each player is allowed to take 1, 2, or 3 marbles from the pile on each turn. A player wins if they take the last marble. If Player A goes first and both players play as well as possible, which player will win, and how? **Justify your answer.**

Problem 2
Player A and Player B are playing a game where there is one pile of N marbles. Each player is allowed to take 1, 2, or 3 marbles from the pile on each turn, with Player A taking the first turn. A player wins if they take the last marble. For what values of N (the starting number of marbles in the pile) can Player A guarantee victory? How can Player A win in these cases? **Justify your answer.**

Problem 3
Player A and Player B are playing a game where there is one pile of N marbles. Each player is allowed to take any number of marbles that is a power of 2 (1, 2, 4, 8, ...); Player A has the first turn. A player wins if they take the last marble. For what values of N (the starting number of marbles in the pile) can Player A guarantee victory? How can Player A win in these cases? **Justify your answer.**

Figure 1: The first three problems in the task sequence.

Our analysis focuses on three groups' work on Problems 1 through 3 (Figure 1). The three groups in our analysis worked on these three problems for a total of 35 minutes, 29 minutes, and 45 minutes, respectively. Because our study focuses on students' generalization and justification practices, we transcribed the segment of each group's work on each of Problems 2 and 3 from the first time a group made a generalization or conjecture about the problem to the time when their work on the problem ended. We then analyzed each group's generalization processes and attempts to justify their conjectures, noting the degree to which each group formalized its reasoning using induction. We adopt the perspective that students' justification attempts offer some evidence of what they consider to be compelling arguments about winning strategies in combinatorial games, and in particular, of what aspects of PMI they are motivated and able to adapt to proving processes in this context.

Results and Analysis

In this section we provide a detailed description of one group's work on Problems 1 through 3 and briefly summarize the work of the two other groups in our analysis.

The Case of Group 1: Evidence of Process Pattern Generalization

Group 1, consisting of Bridget, William, Grace, and Ryan, used the names "John" and "Fluffy" in place of the letters A and B to refer to the two players. They used tables throughout the problem session to represent possible sequences moves in games, and gradually began to use these tables to represent branching cases that could occur within the same game.

After finding in Problem 1 that the first player can win by ensuring that the other player always receives a multiple of four marbles, the group moved on to Problem 2 and began categorizing possible starting positions as winning or losing for Player A. In the group's initial work on this problem, after identifying some examples of initial positions that are winning for Player A, Grace stated the conjecture, "So it's like -- basically, all the numbers except for the multiples of 4, because in that case, Fluffy would win." We interpret this as result pattern generalization, since Grace appeared to obtain this insight from a table of cases the group had considered, and because the group had not publicly justified all of its previous claims about winning positions for the first player.

The group chose to continue considering specific examples to gather evidence for this conjecture. Ultimately, the group returned to its conjecture and finalized it by writing it on the board: "if $N \equiv 1$, 2, or 3 (mod 4), then Player A will win (by generalization of P1). If $N \equiv 0$ (mod 4), then Player B will win." The group then debated whether to provide a written proof of this conjecture; while William remarked that "If we wanted to prove that, we could probably use induction or something," Grace indicated that this was not needed since they had already explained that their conjecture was true "by generalization of the first problem." We interpret this segment of discussion as indicating that the group saw an opportunity to formalize their argument using mathematical induction, but found such formalization to be unnecessary in this case, possibly because of the similarity between the reasoning used in Problem 2 and that used for the more specific Problem 1. We hypothesize that Problem 2 did not create intellectual necessity for PMI for this group; a proof by induction would not have done more to convince this group that their conjecture was universally valid.

On Problem 3, after testing the cases $N \leq 12$, the group correctly conjectured that Player A has a winning strategy if and only if N is not divisible by 3. The group then went on to write an argument justifying this conjecture:

> If 3 [does not divide] N, then Player A will win. If 3 [does not divide] N, then N ≡ 1 or 2 (mod 3), so Player A can leave Player B w/ a multiple of 3. If Player B has a multiple of 3, then B can only remove 1 mod 3 or 2 mod 3 marbles, leaving A w/ 1 mod 3 or 2 mod 3 marbles, so A can win.

Prior to the group's production of this argument, William had stated that each power of 2 is congruent to 1 or 2 modulo 3; this allowed the group to reason that if Player B receives a multiple of three marbles, then any move by Player B will reduce the number of marbles to a non-multiple of 3. In this argument we see evidence of process pattern generalization based on the group's work on specific examples: both the insight that a power of 2 cannot be a multiple of 3 (and that this is important in limiting what a player can do if given a multiple of 3), and the strategy of reducing the number of marbles to a multiple of 3. However, the argument as written does not explain why "A can win" after receiving a smaller number of marbles congruent to 1 or 2 modulo 3 (and one can envision this smaller number being outside of the range of specific examples that the group tested directly).

The Cases of Groups 2 and 3: Obstacles to Process Pattern Generalization

Like Group 1, Groups 2 and 3 correctly determined that in Problem 2, Player A has a winning strategy if and only if *N* is not a multiple of 4. Neither group used induction to ground claims that a first move for Player A would place Player B in a losing position; they instead referenced their prior work on Problem 1 and examples they explored in Problem 2. We hypothesize that the regularity of winning and losing positions in Problem 2 led both groups to the belief that a formal proof by induction was not essential for justification of their conjecture.

While both Groups 2 and 3, like Group 1, arrived at a correct answer to Problem 3, their attempts at justification differed significantly. In attempting to show in general that a multiple of 3 would be a losing position in this game, Group 2 did not take into account possible moves for the next player other than taking 1 or 2 marbles, even though they did account for other possible powers of 2 in the preliminary example work that led to their conjecture. Therefore, while Group 2 was able to render a partial explanation of how Player A could seize a winning strategy by taking 1 or 2 marbles if *N* is not a multiple of 3, their argument did not fully demonstrate that this first move would put Player B in a losing position. Group 3 attempted to prove its conjecture for Problem 3 using PMI, but in doing so, they attempted to establish values of *N* as winning or losing positions by proving the false claim that an integer *N* can be written as either a sum of an even number of powers of 2 or the sum of an odd number of powers of 2, but not both. They embarked upon this strategy despite the fact that expressing an integer as a sum of powers of 2 had not been a key part of their reasoning about specific examples that led to their conjecture.

Discussion and Implications

The results of our interviews are not necessarily indicative of students' ability to use PMI to generalize and formalize recursive reasoning. We did not require students to produce formal proofs during the interviews, so any attempts to use induction or other proof techniques reflected students' desire to confirm or formalize a result, or their sense that we wished for them to do so. In a future study we hope to ask groups of students to work on the same sequence of problems in one task-based interview, then ask them in another interview to write formal proofs of their results.

Nonetheless, we claim that observing students' work on combinatorial game problems provides useful insight about obstacles that may hinder students' efforts to make the cross-cultural translation between the empirical and recursive reasoning that often occurs naturally in exploratory mathematical activity, and the formal inductive justification accepted by the mathematics discipline. First, regularity in the structure of a problem may eliminate intellectual necessity for formal proof by induction. Second, our preliminary results suggest that students may have difficulty translating the recursive reasoning used in specific examples into a proof by induction; and in fact, they may use reasoning disanalogous to their prior reasoning when attempting a proof by induction. Finally, when students do successfully transfer their recursive reasoning into an induction argument, some work may be needed to impress upon students a sense of the importance of a base case, and the function of the structure of an induction proof in providing a foundation on which higher-order cases can rest on lower-order cases.

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