FOUNDATIONAL ALGEBRAIC REASONING IN THE SCHEMES OF MIDDLE SCHOOL STUDENTS WITH LEARNING DISABILITIES

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During the course of a three-month teaching experiment, two middle school students with learning disabilities were found to form operations foundational to algebraic reasoning as they constructed mathematical equivalence schemes. In particular, the students' schemes were considered to be algebraic because they contained the cognitive roots of the distributive property, quantitative unit conservation, and solving linear equations of the form ax=bc with whole number solutions (where a, b, and c are whole number constants). The algebraic character of the students' operating was based on the extent to which the students operated on the structure of their additive and/or multiplicative schemes as they solved tasks that required them to create equivalence between two multiplicative compilations.

Keywords: Algebra and Algebraic Thinking, Number Concepts and Operations, Special Education

Introduction

The purpose of this paper is to demonstrate how the equivalence schemes of students with learning disabilities can be considered algebraic. Additionally, I suggest the algebraic character of such schemes is rooted in how students create and coordinate composite units. This study is grounded in the work of early algebra researchers who have focused on the algebraic character of student's ways of operating prior to a formal algebra course (cf. Blanton & Kaput, 2005; Carraher, Schliemann, Brizuela, & Earnest, 2006; Hackenberg & Lee, 2015). Research in the area of early algebra has become crucial for schools as the number of students who encounter algebra in middle school has increased substantially in the last twenty years.

At the same time, opportunities to be included in grade-level general education have grown for middle school students with learning disabilities due to education policy (Every Student Succeeds, 2015; Individuals with Disabilities Education Act, 2004) (Hord et al, 2019). But access is not enough. Hord et al. (2019) explains that for students with learning disabilities to be successful in middle school and as they progress to formal algebra in high school, they will need to make sense of complex algebraic concepts (Bouck, 2017; National Council of Teacher of Mathematics (NCTM), 2000). This report addresses how the whole number operations of students with learning disabilities can form the foundation for reasoning algebraically and learning complex algebraic concepts. Specifically, I focus on how the cognitive roots for the distributive property, quantitative conservation, and solving linear equations of the form ax=bc (where a,b, and c are whole number constants) can be found in relational equivalence schemes.

Relational Equivalence Schemes and Algebraic Reasoning

The schemes and reasoning I investigated were not formal algebraic schemes, but rather, the multiplicative and equivalence schemes of students that form the underpinnings for formal algebraic concepts. From an ontogenetic perspective, this means schemes are described as algebraic if they can later be reorganized into algebraic schemes (Hackenberg, 2006; Steffe, 2001). In this section, I first describe schemes of mathematical equivalence (Woodward, 2016). I then discuss formal algebraic ideas related to the equivalence schemes: the distributive property, quantitative unit conservation, and solving linear equations.

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Mathematical Equivalence Schemes

Students have the opportunity to construct equivalence schemes when they are asked to produce equality between related multiplicative compilations. For example, a student may be told that person A has 3 baskets of apples with 4 apples in each basket, while person B has 8 baskets of apples with 4 apples in each basket. They are then asked to make it so that each person has the same amount of apples. In this case the two multiplicative compilations differ by the number of composite units (CU) (3CU of 4 singletons and 7CU of 4 singletons), but they also could differ by the size of the CUs (3CU of 4 singletons and 3CU of 5 singletons). There are many solution methods to such a task, including some that indicate a student is engaged in unidirectional thinking as they focus on transforming only one quantity. Others, though, suggest a student is considering relationships between quantities across the compilations as well as notions of balance between them.

I have identified two schemes that students construct while solving tasks like those mentioned in the previous paragraph, a Relational Equivalence (RE) scheme and a Quantitative Relational (QRE) scheme (Woodward, 2016). RE and QRE both incorporate additive balancing operations that support creating equivalence between the two multiplicative compilations. When operating with an RE scheme, a student first multiplicatively produces the totals of 1s from each compilation (3x4=12; 7x4=28 in the example above). They then find the difference in 1s between the totals (28-12=16) and create equivalence by operating on the totals with some or all of the 1s in the difference. For example, they may halve the difference between the two compilations and then re-distribute the 1s (16/2=8; 12+8=28-8).

When a student constructs a QRE, they can operate in the same way, but they can also do more. Instead of operating on 1s, they may choose to focus on the CUs. They can lift their operations on 1s to operations on CUs that require anticipation of multiplicative structures. A student operating with a QRE may first produce a difference in CUs between the two compilations (7CU-3CU=4CU). They then can create equivalence via additive operations on the CU. For example, they may re-distribute the CUs in the difference (4CU/2=2CU, 3+2=7-2) or transform the CU in one of the original compilations to create equivalence (3CU+4CU=7CU).

In this paper I argue that these two schemes are of particular importance because they can provide valuable insight into how students can engage with the distributive property, quantitative unit conservation, and solving linear equation prior to a formal introduction. Moreover, they may be able to provide a path to generate these ideas from students' whole number operating.

Distributive Operations

Researchers have described how students exhibit operations needed for the formal distributive property in algebra, $a(b+c)=ab+ac$ where a, b, and c are real numbers, in whole number (e.g., McClintock et al., 2011; Tzur et al.; 2009) and fractional situations (e.g., Hackenberg & Lee, 2015; Hackenberg & Tillema, 2009). In the case of whole numbers, McClintock et al. (2011) demonstrated a student's use of a distributive operation as the student found the difference between two multiplicative compilations. For example, when presented with two multiplicative compilations such as 19 boxes of candy with 6 pieces of candy in each box and 15 boxes of candy with 6 pieces of candy in each box, a student could anticipate the multiplicative structure of each compilation and identify 4 boxes of candy as the difference. Implicitly included in their mental operations is the underlying quantity of 6 pieces of candy in each box. It is distributed over each set of composite units. The student could mentally anticipate the units-coordination in each of the three sets of composite units without enacting them. For such a student, 19 groups of 6 minus 15 groups of six is the same as 4 groups of 6.

Distributive operations have also been described in fractional reasoning. Hackenberg & Tillema (2009) provided evidence of a student's distributive operation as they multiplied two fractions. The student produced a fractional amount of each unit and then added them together to produce a fractional amount of the sum of the units. For example, when asked to find 1/5 of 4/7 of a mile, a student may first partition 4/7 into 4 units of 1/7 and then partition each 1/7 mile into 5 units. In doing so, they distribute the 1/5 across the 4 units of 1/7 miles. They can then solve the problem by adding 4 copies of 1/5 of 1/7 [1/5(4/7) = 1/5 (1/7+1/7+1/7+1/7)]. Whether the situation is multiplicative or fractional, the key is that the student anticipates the result of distributing one unit across another. Such an anticipatory scheme is considered to be algebraic because it can be reorganized into the formal algebraic idea of the distributive property (Hackenberg & Tillema, 2009).

Quantitative Unit Conservation

Olive and Cağlayan (2008) described *Quantitative unit conservation* as a "coordination of coordinated quantities" (p.271). Quantitative unit conservation emphasizes how quantities found in numerical or algebraic tasks often not only need to be coordinated locally within an equation, but also need to be coordinated across the whole equation. Writing and solving systems of equations, for example, requires schemes constituted by complex relationships that also focus on relating the underlying quantities in the task.

Suppose a student is asked to deposit \$1,000 between two bank accounts from banks A and B, where Bank A provides an interest rate of 4% and bank B has a rate of 6%. If a goal is set to earn 5% interest, two equations could be written and then solved via methods for systems of equations. A possible equation relating the interest earned from each account to the total interest could be $.04x+.06y=.05\text{x}1{,}000$ (x is the amount invested at bank A and y is the amount invested at bank B). However, it is not uncommon for a student to leave off the \$1,000 and only write $.04x+.06y=.05$. The cause of this can be linked to quantitative unit conservation. A student may focus on equating the percentages instead of considering the global coordination of the quantities and the need to equate quantities of money.

Solving Linear Equations

Hackenberg describes how the equation $ax=b$, where a and b are real numbers and x is an unknown, can be thought of as a statement of division (2006). Similarly, the equation $ax=bc$, where a, b, and c are real numbers, can also be thought of a statement of division (once b and c are coordinated multiplicatively). This is not the only way, though, to conceptualize the equation $ax=bc$. For the purposes of my study, the equation $ax=bc$, where a, b, and c are whole numbers, can be thought of as a statement of equivalence. Many researchers have espoused the need for children to view the equal sign as a relational qualifier (e.g., Baroody & Ginsburg, 1983; Kieran, 1981; Knuth, Stephens, McNeil, & Alibali, 2006) when solving linear equations in formal algebra. When a QRE scheme is formed, a scheme of balance is constructed that can be reorganized into a relational understanding of the equal sign.

Furthermore, in the special case where a, b, and c are whole numbers and x is also a whole number, students who do not have fractional reasoning available can still enlist their multiplicative schemes to find a solution if they view the equal sign as a relational qualifier. For example, when solving the equation 4x7=__x14, a student who sees the equal sign as a unidirectional symbol may write 28 or 392 in the blank provided because they multiply the 4 and 7 or all three of the numbers. A student with a relational scheme of equivalence, in contrast, can reason the left side is equal to 28, which is equal to 2 times 14. Moreover, a student with a QRE scheme can reason 4 times 7 is like having 4 groups with 7 in each group, and doubling the group size from 7 in each group to 14 in each group means the number of groups must be halved from 4 groups to 2 groups if equivalence is to be maintained. Whether the unknown is symbolized as a blank space or a letter, a QRE scheme provides a foundation to create meaning for the operations needed to solve the equation.

Conceptual Framework

Examination of students' operations and schemes in my study was grounded in the reflection on activity-effect relationship (*Ref*AER*) framework (Simon et al., 2004; Tzur & Simon, 2004). *Ref*AER* is a refinement of constructivist scheme theory (Piaget, 1985; von Glasersfeld, 1995) that provides a lens for interpreting how a student forms a new conception through two types of reflections on their mental activity (Simon, Tzur, Heinz, & Kinzel, 2004). The definitions of schemes and operations are those described by von Glasersfeld (1995). Schemes are comprised of mental actions, or operations, and consist of three parts. The first part requires an experiential situation; with the caveat that the experiential situation is nothing more than the student's perception of what is offered. As they perceive the situation, the student sets a *goal* as to what activity to engage in. The second and third parts of a scheme are, respectively, the specific mental activity that is called up by the experience (and attached goal) and the student's expected result.

In addition to equivalence schemes, constructs integral to this study were children's multiplicative reasoning and number schemes. The Explicitly Nested Number Sequence (ENS) (Steffe & Cobb, 1988) and the Generalized Number Sequence (GNS) (Steffe, 1994) provide a distinction for the students' operations on CUs. In each, the student can make sense of the nested relationship between compilations with like units. For example, 7 CUs of 4 (7 baskets of 4 apples) and 6 CUs of 4 (6 baskets of 4 apples) are embedded within 13 CUs of 4 (13 baskets of 4 apples). Additionally, a student with GNS can anticipate a multiplicative structure made up of abstract, iterable units (four 1s distributed over each of the 7 CU) prior to operating with it (Steffe, 1994).

Methodology

Over two and a half months, a pair of 8th graders, Joe and Javier, participated as a pair in 14 teaching episodes taught by the author as part of a teaching experiment. The videotaped teaching sessions occurred twice per week and lasted 30 to 45 minutes. A witness-researcher was present for each of the sessions. Data analysis was conducted during the planning and evaluations of teaching sessions and also retrospectively. During the ongoing analysis, critical events were identified, discussed, and used to build second-order models (Steffe & Thompson, 2000) of the students' ways of operating. These models informed decisions as to which tasks and prompts to select for subsequent episodes. The tasks and prompts had a two-fold purpose: to facilitate the construction of new conceptions or to test the anticipatory nature of current conceptions. In the retrospective analysis, video segments and students' written work were used to make inferences about the ways of operating of the students and the algebraic character of their operating.

Joe was selected for the study for two reasons. First, he was attending twice-weekly pull-out sessions for math with a special education teacher. This was because he was identified by school personnel to be a student with a learning disability in reading who needed additional support with areas of math such as solving word problems. The school psychologist identified the Wechsler Intelligence Scale for Children (WISC) and the Wechsler Individual Achievement Test Second Edition (WIAT-II; Wechsler, 2005) as the two main assessments used by the school to identify students with learning disabilities. The second reason Joe was selected was that a base-line assessment revealed he was operating with at least an Explicitly Nested Number Sequence (ENS) (Steffe & Cobb, 1988).

Javier was also purposefully selected for two reasons. The first was his identification by school personnel to be a student with a mild cognitive disability. He received mathematics instruction in a self-contained classroom with a special education teacher. Secondly, Javier was considered to be operating with a Tacitly Nested Number Sequence (TNS) (Steffe & Cobb, 1988). Javier did not operate at the same level as Joe, but he was paired with Joe and part of the study because he was able to multiplicatively coordinate quantities fluently in multiplicative and divisional contexts.

Data Excerpts

In this section I describe how both Joe and Javier enlisted their equivalence schemes to solve a particular task. Javier operated on the structure of his additive schemes as he enlisted an RE scheme (Woodward, 2016). Joe went further and operated on the structure of his multiplicative schemes as he created equivalence using his QRE scheme. During their 10th session on May 15th, Joe and Javier were given the task of creating equality between the following two multiplicative compilations: Javier buys 19 bags of candy with 6 pieces of candy in each bag. Joe buys 15 bags of candy with 6 pieces of candy in each bag. An additional constraint of *no written work allowed until after they had a solution* was imposed by the researcher in an attempt to provoke Javier and Joe to move past initially coordinating the two quantities in each compilation to produce a total of 1s.

To solve the task, Joe thought for only a couple of seconds and then wrote:

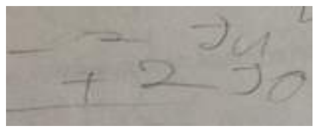

Figure 1: Joe Operates on Composite Units to Create Equivalence.

Unifix cubes were provided to the students in the form of 19 groups comprised of 6 cubes and 15 groups also comprised of 6 cubes. It was common for the students to be asked to demonstrate their solution with cubes after solving the task. At this point I asked Joe about his solution in Figure 1. Joe's response was, "Yeah, I was thinking about bags." Joe then proceeded to demonstrate his solution by moving 2 groups of 6 from the 19 groups to the 15 groups so that each compilation became 17 groups of 6. From Joe's explanation, I inferred he operated on the composite units *between* the compilations. Joe anticipated the multiplicative structures that would be formed if he coordinated the two quantities in each compilation. Such anticipation took the form of a re-presentation to himself of a quantity of composite units with identical sizes from each compilation. Joe was operating with iterable composite units (Steffe, 1994), and so the structures of these two compilations were essentially the same except that there were more composite units in the larger compilation. This understanding enabled Joe to reflect on their relative sizes and to conceptualize the smaller compilation as nested in the larger compilation (15 boxes of 6 nested within 19 boxes of 6). For Joe, the difference in composite units was a way to describe the difference between the two compilations.

Next Joe enlisted his additive schemes as he found the difference in composite units between the two compilations (19-15=4 boxes). His operating to find the difference was predicated on his anticipation that the difference in composite units also described the difference between the total 1s. This anticipation was available to him because the "4 boxes" signified 4 composite units with 6 single units in each that could be coordinated to produce a total of 1s if he chose to enact the coordination. To create equality, Joe increased the smaller compilation (15 boxes with 6 candies per box) by half the difference (4 boxes with 6 candies per box) and decreased the larger total (19 boxes with 6 candies per box) by half the difference. Joe had previously enlisted similar operating, but it was on 1s and not composite units. He re-distributed half the *composite units* in the difference (2 boxes) to the larger compilation (19 boxes) and the other half (2 boxes) to the smaller compilation (15 boxes).

At the end of the task, the students were also asked to write an equation representing their solution. Joe's equation (see Figure 2 below) again provided evidence that the single units (6 cubes per box) were still available to him even though he had operated solely on composite units throughout the process.

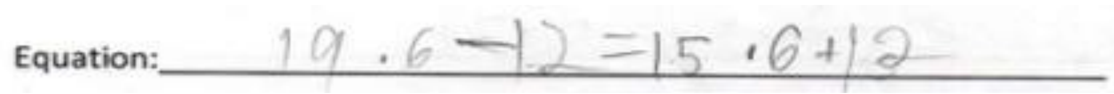

Figure 2: Joe Represents His Solution with an Equation.

It was a requirement for the equations to contain the original quantities from the task. This meant the 6 pieces of candy in each bag needed to be present. When Joe wrote 19 times 6 and 15 times 6, he recognized they represented a quantity of 1s. For the equation to make sense, the two bags from the difference that were re-distributed also then had to be a quantity of 1s. Joe accomplished this by converting the 2 bags of 6 into 12 pieces of candy. Doing so enabled the equation to make sense globally with each representing a quantity of 1s that were equivalent.

Javier also created equality between the two compilations, but he relied on his prowess with multiplying two numbers rather than operations on the multiplicative structures. He first mentally produced the totals (19 boxes of candy with 6 pieces of candy per box=114 pieces and 15 boxes of candy with 6 pieces of candy per box=90 pieces). Next, he added 10 to 90 and subtracted 10 from 114. He produced two new totals of 100 and 104. Javier purposely added a quantity of 1s to the smaller total (90+10) and subtracted a quantity of 1s from the larger total (104-10). I inferred he performed this operation because he anticipated that enacting his additive schemes would bring the two totals closer together by increasing the smaller total and decreasing the larger total. Moreover, when his transformations did not immediately produce equality, Javier continued operating on the smaller quantity to increase it and bring it into balance with the larger quantity. He finished by adding 4 more to 100 so that both totals were now equal to 104. In other sessions, Javier demonstrated that he could continue with more successive operations on both quantities if necessary.

Javier operated on two levels of units, the composite units and the units they contained. Moreover, the composite units were not abstract in nature. This meant he could not *anticipate the results of coordinating the units or the multiplicative structures* they would produce. Thus, he had no opportunity to operate on or with the multiplicative structures. Even when given compilations with large numbers and constrained to mental calculations, Javier computed the totals first and then operated on 1s.

Discussion

In this section I describe how I considered Joe's schemes to be algebraic, while Javier's schemes were algebraic to a lesser degree.

Distributive Operations

Joe operated on CUs as he subtracted 19 boxes of candy minus 15 boxes of candy to yield 4 boxes of candy. Joe mentally anticipated the units-coordination in each of the three sets of CUs involved in the computation as the unit rate of 6 pieces of candy in each box was distributed over each set of CUs. Hence, Joe's mental operations implicitly included the underlying quantity of 6 pieces of candy in each box. He reasoned that 19 groups of 6 minus 15 groups of 6 is the same as 4 groups of 6. I would symbolize this distributive reasoning as 19x6-15x6=(19-15)x6=4x6. Additionally, to produce equality he added two boxes to the smaller compilation and subtracted two boxes from the larger compilation. His reasoning was that 19 groups of 6 minus 2 groups of 6 is the same as 15 groups of 6 plus 2 groups of 6. This reasoning can be symbolized as 19x6-2x6=(19-2)x6=15x6+2x6=(15+2)x6. Again, this representation illustrates the distributive property as 6 is distributed across the quantities.

Joe operated on and with the multiplicative structures of the two compilations provided. His reasoning took on an algebraic character as he enlisted his additive schemes to operate *on* the anticipated multiplicative structure as he found the difference in composite units between the two compilations (19-15=4 boxes). To create equality, Joe also inserted the multiplicative structure of the compilations and the difference into the structure of his equality scheme. The multiplicative structure was readily available to him prior to operating because of his GNS. Joe's reasoning was algebraic because he operated on the structure of his equality scheme *with* the multiplicative structures of the difference (as quantities of composite units) and the existing compilations.

Quantitative Unit Conservation

Joe also exhibited quantitative unit conservation. He demonstrated such reasoning with his equation (see figure 1) and when he described his solution to the ask. After Joe grabbed two groups of cubes from the larger compilation and moved them to the smaller one, he stated, "I take these 2, and then I give them to me. And then they'd be equal. But I just can't say 2. I have to label. So I just put 12." Joe's explanation indicated that he coordinated quantities *within the equation* and *across the whole equation* when generating relationships between the quantities. The key coordinating quantities within an equation and across the whole equation was anticipating the multiplicative structure and operating on the composite units. Joe operated on the composite units (19-2 and 15+2) within the equation. At the same time, he considered the relationship of the multiplicatively coordinated quantities that also contained the unit rate (6 cubes per tower) across the equation. As he demonstrated his solution with the concrete objects (Unifix cubes), the 1s that comprised the composite units were present and available when he needed to include them in his equation. Joe's explanation of his use of the word "two" indicated that he was aware of the need to coordinate the quantities across the equation. Joe reasoned that 19·6-2 was not equal to 15·6+2 because the underlying total of 1s was not equal. As he equated the composite units, he also recognized the totals in 1s needed to be equal. His reasoning was algebraic as it formed the cognitive root of quantitative unit conservation.

Solving Linear Equations

The relational equivalence schemes constructed by both Joe and Javier can serve as a basis for developing a relational understanding of the equal sign. In the discussion above, I illustrated how they both enlisted relational equivalence schemes as they operated on either 1s or composite units simultaneously. This can be contrasted with a student who only operates on one quantity at a time and creates equality by transforming one quantity into the other, a unidirectional scheme of equivalence.

When Javier initially operated on both quantities simultaneously, he purposely added a quantity of 1s to the smaller total (90+10) and subtracted a quantity of 1s from the larger total (104-10). This operation was performed because he anticipated that enacting his additive schemes would bring the two totals closer together by increasing the smaller total and decreasing the larger total. Moreover, when his transformations did not immediately produce equality, Javier continued operating on the smaller quantity to increase it and bring it into balance with the larger quantity. Javier also demonstrated that he was able to do successive operations on both quantities if necessary.

Javier's reasoning was algebraic because he operated with the structure of his additive scheme on the structure of his equality scheme as he operated on both quantities simultaneously. Moreover, he anticipated that the results of his operations could be used for further operating. As Javier created equality via successively smaller transformations, he anticipated a relationship between these transformations and a single, larger transformation. He provided evidence for this anticipation by describing adding 14 to 90 rather than adding 10 and 4. Adding 14 was a single transformation that was a sum of the two smaller transformations of adding 10 and adding 4.

Furthermore, I suggest their RE and QRE schemes form the cognitive roots for symbolizing and solving linear equations of the form $ab=xc$ (where a, b, and c are constants and x is an unknown). Javier could use his relational scheme in conjunction with his operations on tacit composite units to solve *ax*=*bc*. He could multiplicatively produce the total from *ab*, next guess a value for x, then multiplicatively produce the total from *xc* to check to see if matches the total from *ab*. If the totals do not match, he could repeat the process. Without operating with the multiplicative structure of *ab* and *xc*, this would be the primary method available to Javier.

I also suggest Joe could operate with the multiplicative structure of *ab* and *xc* to solve the equation. He could anticipate the multiplicative coordination of the quantities *a* and *b*. He could then reflect on the relationship of this coordination to the multiplicative coordination of quantities *x* and *c*, even though *x* was yet to be determined. He could then solve for *x* via reasoning about *multiplicative relationships* between the multiplicative structures of the compilations *ab* and *xc*. For example, if *c* was twice as large as *b*, Joe could reason that *x* must be half as large as *a*. The key differences between his reasoning and Javier's would be his operations with abstract composite units and his anticipation of the multiplicative structure.

Conclusion

This research demonstrates how the algebraic reasoning of middle school students with learning disabilities is afforded and constrained by their whole number operations. It also provides examples of how students with learning disabilities can operate on and with the structure of their schemes while engaging in complex algebraic reasoning. Finally, this research also supports Hackenberg's reorganization hypothesis (2016) for algebraic reasoning within the context of whole number operations.

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