EYE-TRACKING MATHEMATICAL REPRESENTATIONS – FINDING LINKS BETWEEN VISION AND REASONING

COMPRENDIENDO LAS REPRESENTACIONES MATEMÁTICAS CON MÉTODOS DE RASTREO OCULAR – UNA APROXIMACIÓN BASADA EN LA TEORÍA

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Eye-tracking studies need adequate theoretical frameworks for producing insights about mathematics learning. The current study uses eye-tracking to investigate the effectiveness of tables and diagrams for supporting covariational reasoning amongst elementary students (n = 60). The theoretical framework emphasizes the cognitive functions of representations. Students showed more covariational reasoning around diagrams. The fixations showed that tables concentrated students' attention on the dependent variable data, whereas diagrams distributed students' attention evenly across the numeric and visual elements of the task. According to the theoretical framework, tables did not constrain a covariational interpretation of numerical data, whereas diagrams effectively constrained covariational interpretations, disrupting recursive tendencies and promoting the construction of a mental model of covariation.

Keywords: Representations and Visualization, Research Methods, Cognition, Algebra and Algebraic Thinking

Eye-tracking methods have much potential for studying mathematical thinking and learning. However, more work is necessary for developing conceptual frameworks to guide the design of eye-tracking studies and interpret eye movement metrics (Strohmaier et al., 2020). Here we report research that illustrates an attempt to make conceptual connections between low-level vision mechanisms and abstract mathematical reasoning. The study reported below investigated how external visual representations (VRs) might influence the reasoning of pre-algebraic elementary students while solving tasks that are widely used in the functional approach to early algebra, presented in tabular and diagrammatic formats. Conceptual frameworks of learning with multiple representations allowed us to formulate and respond to research questions with theoretically-driven interpretations of eye movement data.

Generalization and representation in Functional Thinking

The Functional Thinking (FT) approach to early algebra uses the function concept to articulate ideas such as variable, covariation, generalisation, and symbolic notation. FT research algebra relies heavily on tabular tasks to investigate and support generalization processes. The study reported below addresses a task that require students to define missing instances of a dependent variable, e.g., completing missing cells in a function table.

Learners might use at least three approaches for solving tabular tasks (Smith, 2008): (1) Recursive patterning involves attending to variation in sequences of values, (2) a covariational approach means analysing simultaneous change in two or more quantities, and (3) a correspondence approach emphasizes the relation between pairs of variables. The use of covariational and correspondence approaches indicate a notion of function based on covariational reasoning (Thompson & Carlson, 2017), which entails the conception of two quantities varying simultaneously with an invariant relationship between the values of the quantities, and every value of one quantity determines exactly one value of the other quantity.

Most elementary students struggle to transit from recursive approaches to covariational reasoning while working with tables (Tanışlı, 2011; Wilkie, 2016). The focus on recursion might result from an interaction between a natural the tendency to seek univariate patterns and the visual properties of tables. Consequently, other visual representations, such as diagrams, could ease covariational reasoning by disrupting recursive tendencies. However, research about representational factors in functional thinking is scarce, so the question remains open: How do different visual representations influence students' reasoning during functional thinking tasks?

Approaches to learning with multiple representations

We consider tabular tasks and diagrammatic tasks as cognitive tools that display information to achieve mathematical insight, and not necessarily to depict mathematical objects (Giardino, 2017). Therefore, we draw from the Design, Functions and Tasks (DeFT) framework to learning with multiple representations by (Ainsworth, 2006), which addresses the learning potential of multi-representational systems from a design perspective, considering representational features such as modality and number of representations as design parameters, as well as other dimensions to analyse the effectiveness of multiple representations, namely tasks and functions.

Tabular tasks and diagrammatic tasks are equal in design because both are in the visual modality and combine text with VRs. Tables and diagrams are also equivalent in the task dimension because these representations are common in the elementary classroom and, therefore, students know how to "read them". Pre-algebraic elementary students learn to relate tables or diagrams to the functional thinking domain while working in functional tasks. However, tasks and diagrammatic tasks are different in the functions dimension.

The DeFT framework outlines three cognitive functions of multiple representations. *Complementary functions:* In multi-representational tasks, the representations should complement each other by differing in the information each contains and the processes that each support. *Constraining functions:* Multiple representations help learning when one representation constrains the interpretation of a second representation. VRs can constrain text because text is ambiguous and VRs are specific (Schnotz, 2005); *Constructing functions:* Multiple representations support effective learning when learners integrate information from representations to achieve insights that could be difficult to achieve with only one representation. In tasks that include text and VRs, each representation is processed by parallel mechanisms resulting in complementary mental models that are mapped onto each other (Schnotz, 2005), thereby extending current knowledge and facilitating deeper understandings.

Representational functions of tabular tasks

Tables are semi-graphical representations that support learning by arranging information to exhibit facts or relations in a compact manner, and by directing attention to unsolved parts of a problem (Cox & Brna, 1995). In the tabular functional tasks reported in the literature (e.g., Tanışlı, 2011), graphic components such as cells, rows and columns, comply with the complementary function by representing mathematical relations that texts cannot represent. For example, columns represent variables, rows represent ordered pairs, and empty cells represent missing instances of the dependent variable. Tables comply with the constraining function by positioning numbers in a way that constrains their interpretation from a covariational reasoning perspective. The constructing function happens when the processing of numerical data produces a mental model of quantities, and the processing of the visual layout produces a mental model of covariation. The mapping of these mental models prompts insights about the invariant rule governing the relationship between quantities.

Representational functions of diagrammatic tasks

Diagrams represent objects with pictorial components that express conceptual relations spatially, and can be idealized or instantiated in some context (Belenky & Schalk, 2014). Diagrams give fast access to meaning, facilitate the comprehension of complex information, and elicit previous knowledge (Tversky, 2011). Diagrams effectively show physical layouts and how things work or are put together, organize information, make abstract ideas concrete, and allow the use of spatial skills (Winn, 1991). We have made explorations with diagrammatic functional tasks of shadow-casting phenomena (Xolocotzin et al., 2018). In these tasks numerical data is complemented by pictorial components representing a pole and its shadow, making an explicit representation of covariation and correspondence between the quantities pole height and shadow length. The pictorial components constrain a relational interpretation of numerical data. For example, the pictorial representations of the pole and its shadow are visually connected, facilitating the interpretation of numerical data from a covariational reasoning perspective. Diagrammatic tasks might comply with the constructing function by facilitating the integration of a mental model of quantitative properties extracted from numerical data, e.g., variation, with the mental model of shadow-casting phenomena, which is relational by nature.

Previous paper-based study

Before the eye-tracking study, we assessed the effects of diagrammatic and tabular tasks with a paper-and-pencil study conducted with 1145 students in Grade 4, Grade 5 and Grade 6, recruited from 16 public schools located in central Mexico. Because the schools are public, they must follow the official mathematics curriculum, which does not include algebraic content. We studied different representational versions of a functional task that required students to identify missing instances of a dependent variable. The task presents two number sets. The first set has 6 numbers of the independent variable. The second set has 3 known numbers and 3 unknown numbers of the dependent variable. Students must figure out the rule governing the relationship between the two variables to identify the missing numbers. There were four options to choose from: (1) functional, (2) recursive, (3) First instance, which is consistent with a rule that only applies to the first pair of data, thereby denoting lack of generalization, and 4) random, which presents three numbers defined randomly. Four versions of the tasks were generated by manipulating two factors: Representation (table or diagram), and context (with context or without context), see Fig. 1. The diagrammatic tasks, either with context of without context, generated more functional responses than tabular tasks. However, this effect was larger in the contextualized version of the diagrammatic task. We also observed that Year 5 students were the most sensitive to the effects of context. We considered these results as evidence that diagrams ease covariational reasoning.

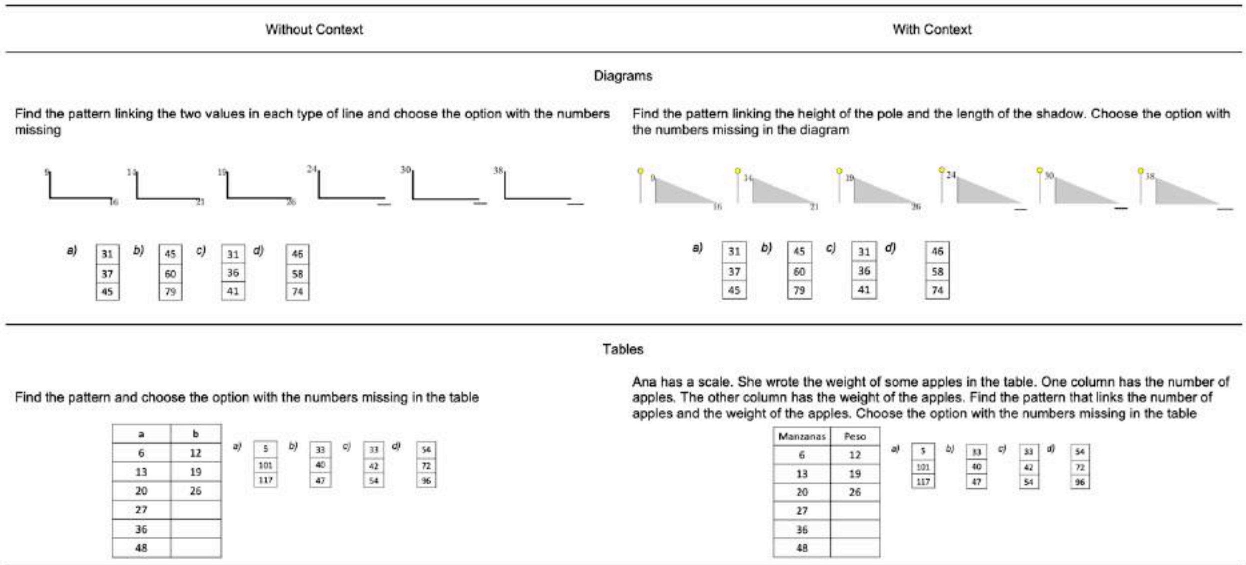

Figure 1 Examples of the tasks employed in the paper-based study.

Overview of the current study

The previous paper-based study suggested that diagrammatic tasks are more effective than tabular tasks at easing covariational reasoning. Albeit informationally equivalent, diagrams seemed to facilitate retrieval of functional information. In line with the DeFT framework, we hypothesized that diagrams are more effective than tables for complying with the functions of multiple representations, which opened our research question: How do diagrammatic tasks and tabular tasks comply with functions of multiple representations such as constraining and constructing? To answer this question, we analyzed students' eye movements to gain insights about the effects of tables and diagrams on students' attention.

Method

Participants

A total of 60 students in Grade 4 (n = 20), Grade 5 (n = 20) and Grade 6 (n = 20) from a public elementary school participated in the study. All students participated on a voluntary basis, with informed consent from parents and school authorities. Three participants failed to reach accuracy levels due to unforeseen circumstances, e.g., spectacles not allowing registration of the participants' eye. Therefore, their data were discarded, leaving a sample of 57 students.

Apparatus and stimuli

The data were collected with a portable eye-tracker Tobii Pro X2-30, with a 30 Hz sampling rate, 0.4 precision (binocular), and 0.32 gaze precision (binocular). Both eyes were tracked. This model allows robust detection of individuals' eye movements, even with unrestricted movement. The eye-tracker was mounted below the screen of a Dell Inspiron 5000 15 inch laptop, which display was set at 60 Hz refresh rate and 1366 x 768 resolution. The distance between the eye-tracker and the edge of the table was held constant at 60 cm.

The stimuli were a series of functional tasks presented in either tabular or diagrammatic format. The tasks were replicated from the "with context" of the previous study (See Fig. 1). Tabular tasks were grounded on a situation involving apples and their weight. The diagrammatic tasks were grounded in a shadow-casting situation involving the height of a pole and the length of its shadow cast. There were 12 tabular tasks and 12 diagrammatic tasks. In each type of task, there were four items involving sums, four items involving subtractions, and four items involving multiplications. The same as in the previous study, each task required the identification of a relationship between two quantities, and selecting one of four response options: functional, recursive, first instance, or random.

Experimental design

The experiment employed a factorial design involving the within-subjects factor representation (table/diagram) and the between-subjects factor Grade (4/5/6). The factor representation was counterbalanced within each Grade. The tasks were presented in fixed order: sums, substractions, and multiplications.

Procedure

Students were tested individually in the school IT suite. They were instructed as follows: We would like you to please help us solving a task. It is important that you know that the results do not have any relation with your grades. Do you have any question? the results are very important for us because we are studying how students solve some mathematics activities. Please, pay attention and do your best effort.

Results and discussion

The behavioural results replicated the previous study, that is, students chose the functional response more in diagrammatic tasks [F (1, 56) = 4.038, $p < .05$, $\eta^2 = .022$]. The eye tracking data, interpreted under the DeFT framework, allowed us to explain this result. Eye-trackers produce a range of eye-movement metrics. We wanted to know which areas components of tables and diagrams were more noticeable for students. Therefore, we used fixation time, which indicates difficulty in extracting information, or that the object is more engaging in some ways (Poole & Ball, 2006). We defined a series of analogous areas of interest (AOIs) corresponding with key components of the tasks, see Fig. 2. One set of AOIs contained data; A1 and A2 show contained the first and second half of the independent variable. A3 contained the first half of the dependent variable, and A4 contained the unknown second part of the dependent variable. A second set of AOIs contained contained the response options, functional, recursive, instance, and random.

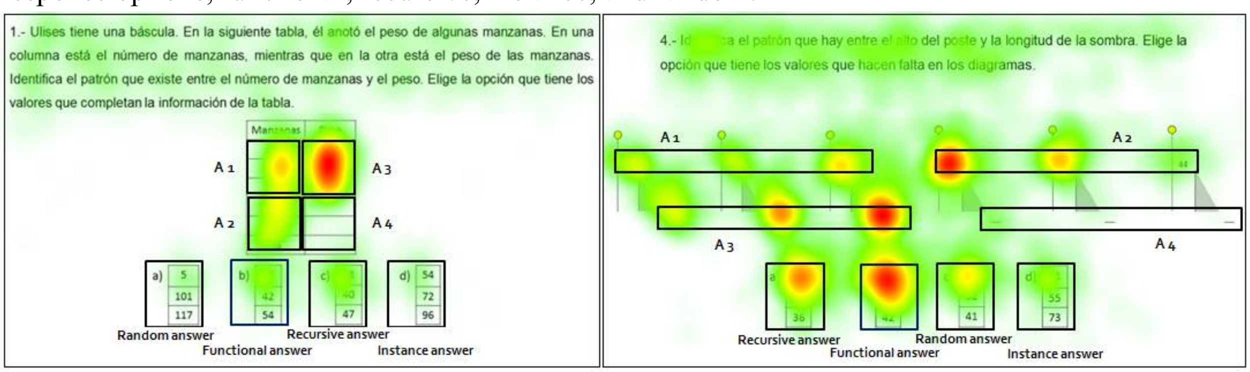

Figure 2 Ares of Interest in Tabular tasks and Functional tasks with an overlay heatmap of fixation duration

The analysis of the data AOIs revealed that students fixated more on AOI3 while solving tabular tasks, which contained the known numbers of the dependent variable, whereas in diagrammatic tasks students fixated evenly across the data AOIs [F (3, 162) = 20.541, $p < .001$, $\eta^2 = .085$]. As for responses AOIs, students fixated more on functional responses while solving diagrammatic tasks [F (3, 162) = 40.209, $p < .001$, $\eta^2 = .108$]. The DeFt framework allows an interpretation of these results. Figure 2 illustrates how students engaged more with AOI A3, which indicates the spatial layout of cells and columns, fails comply with its intended function of constraining a covariational interpretation of numerical data. Therefore, students are unable to integrate the mental model of quantitative properties extracted from the data, with a mental model of covariation. Therefore, they did not achieve the necessary insight for inhibiting the tendency to look for recursive patterns. This might explain why students engaged equally with recursive and functional responses. In contrast, diagrams disrupted the tendency to focus on recursion, and made students to expand the breadth of their attention, and engaged equally with data and probably other elements of the task. This might indicate that they considered all sources of pictorial and numeric information, moreover, pictorial components seemed to constrain a covariational interpretation of data. We argue that diagrammatic tasks allowed the construction and integration of a mental model of quantitative properties extracted from numerical data, and a mental model of a relational situation, extracted from the graphic elements of the task. In this way, students gained covariational reasoning insight and, therefore, engaged more with functional responses.

General conclusion

Eye-tracking research in mathematics education is growing steadily (Strohmaier et al., 2020). However, the release the full potential of these methods for gaining insights about mathematical learning, it is necessary to use theoretical frameworks that allow plausible interpretations of eye-movement data. The presented study aimed to illustrate the benefits of theory-driven interpretations of eye-movement data.

In our first paper-based study, we found that diagrammatic tasks were more effective at easing covariational reasoning than diagrammatic tasks. However, this result could not be explained from paper-based data. So, we had the output but were unable to empirically explain the process leading to such output. We addressed this issue with eye-tracking methods because the cognitive mechanisms involved in learning from visual representations cannot be observed directly. Moreover, these mechanisms rely heavily on unconscious vision processes which operation cannot be intentionally controlled by individuals.

The DeFT framework allowed us to make theoretically-informed accounts of the ways in which tables and diagrams are expected to support covariational reasoning. By analyzing tabular tasks and diagrammatic tasks under the DeFT framework, we identified that these representations were similar in the dimensions design and tasks, but different in the functions dimension. Therefore, we hypothesized that diagrams were more effective for supporting covariational reasoning in the first study because this representation complied more effectively with functions such as complementing textual information, constraining textual information, and constructing insights.

The behavioural results replicated the paper-based results, diagrammatic tasks were more effective for supporting covariational reasoning. The patterns of fixation duration confirmed our hypothesis. Diagrammatic tasks distributed the individuals' attention evenly across the visual display of the task, and directed their attention to recursive and functional responses evenly. In contrast, the tabular task concentrated individuals' attention on the first part of the dependent variable, and directed their attention to recursive only.

An interpretation of results from a cognitive load framework would have been problematic. Diagrams should have produced less functional answers because they have more information and require more cognitive resources than tables. The DeFT framework offered a more parsimonious interpretation of these results. The layout and structure of tables seemed unable to constrain a covariational interpretation of textual data such as task instructions and numerical data, thereby favoring a recursive interpretation of the data. In contrast, diagrams effectively constrained a covariational interpretation of data, disrupting the natural tendency to seek recursive patterns, and allowing the production and integration of a mental models of data's numerical properties with a mental model of covariation.

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