REFLECTIONS ON DIGITAL TECHNOLOGIES IN MATHEMATICS EDUCATION ACROSS CULTURES

Celia Hoyles
UCL Institute of Education, UK
c.hoyles@ucl.ac.uk

Carolyn Kieran
Professor Emerita, Université du Québec à Montréal, Canada
kieran.carolyn@uqam.ca

Teresa Rojano
Center of Research and Advanced Studies (Cinvestav), Mexico
trojano@cinvestav.mx

Ana Isabel Sacristán
Center of Research and Advanced Studies (Cinvestav), Mexico
asacrist@cinvestav.mx

María Trigueros
Instituto Tecnológico Autónomo de México (ITAM), México
trigue@itam.mx

In this plenary presentation paper, we reflect on issues related to the role, potential and extent of integration of digital technologies in mathematics education, and attempt to give our perspectives on these issues from across cultures and over time stretching over the past four decades. Three experts, from three different countries, give their reflections on how mathematics teaching and learning has changed and developed since digital technologies were introduced in schools. Their individual narratives are then complemented by a discussion of the differences in adaptation to the constant changes in technology, changes that arise from diverse socioeconomic, political and cultural visions of the role of digital technologies and their influence and on how the potential of these technologies can actually be harnessed.

Keywords: Technology; Instructional activities and practices; Culturally relevant pedagogy; Computational thinking

Introduction, by Ana Isabel Sacristán and María Trigueros

For this plenary presentation, we invited three renowned experts on the integration of digital technologies in mathematics education: Celia Hoyles from the UK, Carolyn Kieran from Canada, and Teresa Rojano from Mexico. We asked each of them to reflect on what has happened with the roles and potentials of digital technologies in mathematics education since their adoption in school systems, in order to develop a perspective of this topic across cultures and countries.

This paper presents their contributions complemented by our own reflections. The next sections present the experts' contributions. First, Hoyles presents a reflection on how mathematics teaching and learning has changed and developed since digital technologies in general and programming, in particular, were introduced into school mathematics education, specifically in England. Then Kieran, through the lens of the Canadian experience, presents a discussion of how issues related to the emergence of new technologies and the restrictions involved in their use by teachers in their classrooms created conditions that privilege some approaches; she also presents an interesting discussion on how computational thinking relates to mathematical practice and thinking. In the next section, Rojano begins by discussing some results of an Anglo-Mexican project that highlighted some differences in the use of technologies for algebra, then describes some national projects developed in Mexico, and how economic and political circumstances had an impact on their possibilities of generalization and success.

We finish by including our own reflections, based on their visions, on how digital technologies have played an important role in mathematics education and on how different cultural and economical conditions have a strong influence on how the potential of these technologies can actually be harnessed.

In: Sacristán, A.I., Cortés-Zavala, J.C. & Ruiz-Arias, P.M. (Eds.). (2020). *Mathematics Education Across Cultures: Proceedings of the 42nd Meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education*, Mexico. Cinvestav / AMIUTEM / PME-NA. https:/doi.org/10.51272/pmena.42.2020

Reflections on the Role of Digital Technologies in Mathematics Education: insights from the past and ongoing research, by Celia Hoyles

The invitation to make this contribution at PME-NA has provoked me to look back on about 30 years of research in the area of digital technologies and mathematics education, in general and in particular around programming. My passion is and has been to help learners open windows to mathematical knowledge by using digital technologies in innovative, future-oriented and intellectually rigorous ways (Hoyles, 2018). This passion I have shared with Richard Noss with whom I have worked on numerous research projects over the years. The research questions we have posed, the technologies designed, and the research methods employed have all reflected this desire to widen access to learning through digital technology and to tease out the conditions necessary for this to happen successfully. Mathematics is central to the school curriculum, yet all too often mathematics does not engage learners who do not discern the point of the mathematics they are forced to learn. This is important at the individual level, but at the same time, the technology-based 'information society' needs model-based reasoners who can exploit mathematical ways of thinking to make sense of their world. This has become ever more apparent at the time of writing this paper when the world is enduring a global pandemic where everybody is assailed by data and graphs of the numbers infected by Covid 19. These representations and the models that generate them need to be interpreted. Thus we need to take seriously the design challenge to engage our students in school in mathematical thinking and application which as far as is possible is actually needed for them to achieve the goals that they find compelling.

Background

My work has been grounded in constructionism originating from the vision of Seymour Papert (Papert, 1980a), which asserts that one way to achieve sense-making for learners was for them to take the role (to some extent at least) of *producers* rather than *consumers* of digital tools, so they are better able to explain the *effects* of the tools (see for some example Confrey et al., 2009, p. 19). Our approach originated with a Piagetian basis but evolved to embrace "a hybrid social constructivist/sociocultural approach… with a vision of human–machine interaction and design for mathematical activity" (Monaghan, Trouche and Borwein, 2016, p. 10). In my view, fundamental to constructionism are two notions: *epistemological pluralism*, that is accepting the validity of multiple ways of knowing and thinking (following Turkle and Papert, 1992) and *designing for interaction in microworlds*.

I make a small diversion to give the reader a glimpse of the challenge Seymour Papert set us many years ago on this first issue of *epistemological pluralism*. As a member of a plenary panel at a conference in Newcastle, Australia, to commemorate the work of Jon Borwein, I was tasked to address the new ways people think, move and feel mathematically, thanks to the opportunities offered by digital technologies: the abstract went on to state:

> …the emergence of new digital technologies and new theories have helped researchers recognise the breadth and depth of that change and simultaneously provide a framework for the design and implementation of computational tools for learning mathematics. The possibility of putting mathematical objects into motion, for example, fundamentally changes the nature of these objects, how they are perceived and reasoned about; moving these objects changes the bodily actions and gestures of both learners and teachers; making the objects transform, collide and overlap, changes the stories that can be told about them. Research on the use of digital technology has also provided an extraordinary 'window' on mathematical meaning making, to use the metaphor provided by Celia Hoyles and Richard Noss, in part because of the visibility of thought, motion and feeling enabled in expressive digital technology environments. (Drijvers et al., 2016).

The example I gave at the panel was stimulated by Papert's keynote at 10th conference of PME in London way back in 1986 that I simply mention here to provoke the reader. Papert talked about what is generally known as the "alternate segment theorem" (p. 3) and argued how simple it was *if only one thought of oneself in the motion* and then "it is easy to see the total turn of a circular arc is the same as the angle at the centre of the circle"(end of second paragraph, Papert 1986, p. 3).

And what about *microworlds*? In Hoyles (1993), I described the evolution of the *microworld* idea from its genesis in the artificial intelligence community, in which it was used to describe a relatively simple and constrained domain where computational systems could solve problems, to a more broadly conceived environment that served as a concrete embodiment of a knowledge domain or structure. The structure comprises tools that are *extensible* (so tools and objects can be combined to build new ones), but also *transparent* so their workings are visible, and rich in different representations. There is a duality here: a successful microworld is both an epistemological and an emotional universe, a place where powerful mathematical ideas can be explored; but explored "'in safety"', acting as an incubator both in the sense of fostering conceptual growth, and a place where it is safe to make mistakes and show ignorance. And, centrally these days, it is a place where ideas can be effortlessly shared, remixed and improved (for an earlier discussion of these twin aspects of engaging through building and sharing, see Noss and Hoyles, 2006).

Programming and Mathematics

Now let us fast-forward to 2014 where, at least in England, programming or coding is widespread and moreover part of a *compulsory* curriculum for all students from age 6 to 16 years. It is important to recall now that teaching of computer programming is not new, with educational programming languages such as Logo and BASIC widely used in both primary and secondary education settings during the 1980s and 90s. However, Manches and Plowman (2015) highlighted that recent discussion around how to teach programming in schools often omitted the earlier research conducted during this time: some of which are reported in Hoyles and Noss (1992). We need to learn from these earlier studies. Research on the efficacy of programming had in fact produced mixed results (Clements, 1999; Voogt et al., 2015). From the point of view of learning programming per se, some of the key challenges identified were difficulties with programming syntax, dealing with error messages along with the severely limited access to technology within the classroom (Resnick et al., 2009; Lewis, 2010). Since then there have been hugely significant developments in novice programming languages that have overcome many of these challenges (though maybe raised others), and technology access has become increasingly commonplace within schools with resources readily available for sharing through the Web.

When computing became mandatory in English schools in 2014[1], the time was ripe to revisit research on the impact of programming on mathematics learning taking account of the new context and critically learning from the past that had shown the importance of design of microworlds and critical need to take account of the teacher's role. Richard Noss and I then embarked on a new research project, the ScratchMaths (SM) project, which set out to explore the potential of programming for the 9–11 primary1 age group (upper Key Stage 2 (KS2)) in light of the curriculum changes and the renewed enthusiasm and motivation for the teaching of programming in schools. The project consists of a design phase following which the intervention comprising detailed computer tools and curriculum materials is implemented over 2 years with the same pupils. Some factors that contributed to Logo and other early programming initiatives not fulfilling their potential have been mentioned earlier, but a key factor being identified by Noss and Hoyles (1996) as the importance of

[1] During the late 1990s programming faded from the English curriculum and became subsumed and eventually replaced entirely by the subject of ICT (Information and Communications Technology), which focused more on the use of technology than on its creation (Brown et al., 2014).

fostering a *sense of teacher understanding and ownership* of any programming innovation. In addition, there have been significant technological developments since this early teaching of computer programming, with a number of block-based languages such as Scratch now freely available and widely used. These environments have helped to address some of the difficulties of mastering programming syntax, but there remains the challenge of ensuring that teachers first appreciate why they are introducing programming as part of mathematics – and then have opportunities to develop appropriate skills to teach programming.

The ScratchMaths (SM) designed a 2-year intervention aiming to develop the mathematical knowledge of pupils (aged 9-11 years) through programming. The SM approach was to select and design activities around core computational ideas that would then be used as vehicles to explore specific mathematical concepts and promote mathematical reasoning. This approach enables parts of computing to be taught within or as a supplement to mathematics lessons. For a summary of the SM design research phases, activities and outcomes see (see Benton, Hoyles, Kalas & Noss, 2017, for a detailed account of the design of the study). The SM content was divided into six modules, or in terms for this paper, microworlds, three per year. In the first year for 9-10 year-old students, computational concepts (see for example, Wing, 2006) were foregrounded with mathematical ideas more implicit in microworlds titled *Tiling Patterns, Beetle Geometry* and *Collaborating Sprites*. In the second year, the same students (now 10-11 years old) were introduced to mathematical concepts and mathematical reasoning explicitly through a programming approach along with a set of new computational concepts in microworlds titled Building with *Numbers, Exploring Mathematical Relationships,* and *Coordinates* and *Geometry*. SM was intended to comprise approximately 20 hours teaching time across each of these two school years.

Given the challenge of implementing a new curriculum, the SM teachers were provided detailed guidance for navigation through support materials, which were themselves carefully structured and progressive. Prior to teaching each year of the SM intervention, teachers were offered two full days of professional development, spaced a few months apart. During these sessions, the teachers were introduced to Scratch and the SM curriculum content. The SM design was framed by constructionist theory whereby pupils would engage with the mathematical ideas by building programs to explore so the PD followed this approach (see Noss & Hoyles, 2017, 2019). It was then intended that this constructionist approach would be operationalized in the classroom through what we called the '5Es pedagogical framework' with teachers given the opportunity to participate in activities that incorporated exemplars of the different pedagogical strategies. The five unordered constructs of the framework are summarised below:

Explore: Pupils should have opportunities to explore different ways of dealing with constraints and ambiguity as well as investigating their own and others' ideas and debugging different types of errors.

Explain: Pupils should have opportunities to explain their own ideas as well as answer and discuss reflective questions from the teacher and peers.

Envisage: Pupils should predict outcomes of their own and others' programs with specific goals *prior* to testing out on the computer.

Exchange: Pupils should have opportunities to share and build on others' ideas as well as justify their own solutions.

bridgE: Pupils should be helped to make links between contexts beyond the Scratch programming environment by explicit re-contextualization and reconstruction within the language of mathematics, by for example unplugged activities.

The SM intervention was subject to cycles of iterative design research following which it was scaled out across England. The project was also evaluated through a randomised control trial conducted by another university (Boylan et al., 2018[2]). For the trial, 111 English primary schools (6300 pupils) were recruited and randomly assigned to control and treatment groups with the final quantitative outcome measures based on scores in first a test of computational thinking (designed by the evaluation team) to be applied in the first year of the evaluation, and second, scores on national standardised mathematics tests taken by all students at the end of primary school.

For completeness, I summarise the findings of the external evaluation of SM, which reported:

- a positive and significant impact on Y5 Computational Thinking skills, which was
  - particularly evident among disadvantaged pupils that is those who had or currently have free school meals
  - with no difference between girls and boys
- no evidence of impact on the national Key Stage 2 Maths test

Clearly what is interesting in these outcomes for researchers is to seek to explain the reasons for these outcomes. To be honest we are not sure and welcome research replications and adaptations that are underway, which throw light on these issues. I note for example the nationwide large scale study in Spain (INTEF, n.d.), which reported the following: "…the results show that it is possible to include programming activities in 5th grade in the area of mathematics, so that students not only learn to program and engage in computational thinking, but also improve the development of their mathematical competence greater than their colleagues who have worked in this same area using other types of activities and resources not related to programming."

Here I simply mention what I see as important contextual influences in England that might well have shaped the outcomes. First, why the significant positive effect of the SM intervention on CT scores as measured by the test used at the end of the first year of the trial. We cannot be certain, but simply point to the fact that the SM package is a systematic, progressive research-based curriculum that offers detailed support to teachers. The independent evaluators also remarked that SM was popular among the teachers who sustained their participation. Our surveys told us that fidelity to SM in terms of engagement in professional development, provision of SM curriculum time and coverage was very high in Year 5.

However, there was a dramatic drop in this fidelity in Year 6, along with huge variation in pedagogy. This was, we conjecture, a result of the negative impact of the high-stakes testing in mathematics in England at the end of Year 6. There is massive pressure on teachers who therefore might feel unable to engage with a new curriculum. In addition, our data indicated that many SM Y6 classes were taught by *teachers with little or no experience of SM,* either from teaching SM in Y5, or from engaging in the SM professional development (for more detail see Noss et al., in press). This was a matter of fidelity but also of the considerable teacher churn in schools – the average for the staying in the profession is 4-5 years. However, I would go so far to say that where professional learning was taken seriously by schools, as in the first year of the innovation, implementation tended to be successful. And, conversely without this it is very unlikely that an innovation like SM would operate in the classroom as planned.

Celia Hoyles' final remarks

I end by calling for more intensive and systematic classroom research and evaluation to explore the classroom implementation of SM or other computing initiatives, not least as computing is now embedded in school practice, and learners and teachers are more confident and competent in

[2] The evaluation report can be found at https://educationendowmentfoundation.org.uk/projects-and-evaluation/projects/scratch-maths/
The student and teacher materials are freely available from the UCL website http://www.ucl.ac.uk/scratchmaths (creative commons license)

programming. Such research would also need to reconsider methods of evaluation of research outcomes in terms of teacher and pupil learning.

Digital Technologies in Canadian Mathematics Education, by Carolyn Kieran

My contribution to the cross-cultural digital-technologies panel focuses on three aspects: i) how the digital-technology culture has evolved in the Canadian school system since the 1980s when the Logo movement began, ii) brief comments on the characteristics of computational thinking and how they relate to mathematical practice and mathematical thinking, and iii) Canadian research that illustrates the use of digital technologies for fostering mathematical thinking.

Evolution of the digital-technology culture in Canadian schools since the 1980s

When the Logo movement began in the 1980s, a corps of enthusiastic Canadian mathematics educators and teachers adopted MIT professor Seymour Papert's vision of having children use computers as tools to think with. The Logo programming language was at the heart of this movement. The mathematical connections associated with the Logo movement spilled off the pages of Papert's (1980a) book, *Mindstorms: Children, computers, and powerful ideas*. Mathematics education researchers across Canada, as well as in other countries, developed projects involving primary, secondary, and tertiary level students, which were aimed at exploring mathematical ideas in turtle geometry by programming with Logo.

However, the Logo movement in the public schools was hampered by a lack of funds – in other words, the absence of political will on the part of government to invest massively in any such endeavours – that would have allowed for obtaining the appropriate hardware and software. The promising results that were being highlighted in research reports and in presentations at various Logo conferences throughout the 1980s and early 1990s would unfortunately not lead to more widespread implementation in Canadian schools. By the end of the 1990s, many had let go of their interest in Logo. Handheld calculators with graphing capability, as well as Computer Algebra System (CAS) calculators, proved to be easier (and more economically feasible) to integrate into mathematics classes than the more expensive computers necessary for Logo.

Some time passed. Then by the end of the first decade of the new millennium, the situation with respect to digital technologies had begun to change in many ways. In 2006, Jeannette Wing, herself a computer scientist, wrote a brief paper titled *Computational thinking* where she argued that computer science was more than just programming; it also involved ways of thinking (Wing, 2006). The effect of this paper was probably more influential than Wing had expected or even dreamed. In contrast to what had not happened during the Logo movement, the idea of computational thinking began to have an effect on K-12 education. As in other countries, Canadian policy makers and curriculum leaders decided that our students needed to develop their technology skills within the school system.

Education in Canada is not a federal matter; there is no national curriculum. Every province is responsible for setting its own school programs. So far, five provinces have begun to develop new technology programs. The pioneers in this movement have been British Columbia, New Brunswick, Nova Scotia, Quebec, and most recently (in 2020) Ontario. But, in the main, these new programs are stand-alone technology programs – primarily for grades 6-8, with an emphasis on a range of technological tools, processes, and applications, especially coding. In contrast, Ontario opted to integrate coding within its Grades 1-8 mathematics curriculum.

With the exception of Ontario, one wonders if this new interest in computational activity will make its way into mathematics classes? Any optimism one could have might be tempered by the results of a recent survey of teachers of 5- to 14-year-olds in 23 countries around the world, carried out by Rich and colleagues (2019): While the teachers noted that their students loved the new coding activities, they also stated that they were more confident in their ability to teach students coding/computing as a stand-alone subject than they were with integrating it into other subjects. This finding suggests that, along with current curricular changes, an equally important component that cannot be neglected is the need for an ongoing form of professional development to allow teachers of mathematics to keep up with constant changes in digital technologies and to feel confident in their ability to integrate these technologies into the exploration and development of students' mathematical thinking.

Computational thinking, mathematical practice, and mathematical thinking

Papert (1980a) emphasized that when children use computers as tools to think with, they are also "talking mathematics" (p. 6) to these computers. It is upon his shoulders that the present computational thinking movement stands. So it seems appropriate to ask at this point how computational thinking has been characterized in this recent movement. Wing (2006), for example, states that she has drawn on ideas fundamental to computer science and asserts that "computational thinking is using heuristic reasoning to discover a solution" (pp. 33-34). And in a later paper, Wing (2014) writes: "The most important and high-level thought process in computational thinking is the abstraction process" (p. 1). Relevant to the questions addressed to this panel, she adds that computational thinking can be defined as "the thought processes involved in formulating a problem and expressing (with a linguistic representation) its solution in such a way that a computer – human or machine – can effectively carry it out" (p. 1). Interestingly, Andy diSessa (2018) – one of the two authors of *Turtle Geometry* back in 1981 – has taken issue with this point and has argued that non-computer scientists rarely map out exactly how a problem can be solved before actually doing the solving. But is he right?

In opposition to diSessa, and more in line with Wing, Al Cuoco (2018) in a paper on mathematical practice offers three examples. The first of these (see Fig. 1) relates to Wing's emphasis on the process of abstraction and her point about formulating a problem and expressing its solution in a way that a computing being or machine can carry it out.

- I take a guess, say 1200 miles.
- Check it:
  (*) $\frac{1200}{60} = 20$
  (*) $\frac{1200}{50} = 24$
  (*) $20 + 24 \neq 36$
- That wasn't right, but that's okay— just keep track of your steps.
- Take another guess, say 1000, and check it, making the check more efficient:
  $$\frac{1000}{60} + \frac{1000}{50} \stackrel{?}{=} 36$$
- Keep it up, until you get a "guess checker"
  $$\frac{\text{guess}}{60} + \frac{\text{guess}}{50} \stackrel{?}{=} 36$$
- The equation is
  $$\frac{x}{60} + \frac{x}{50} = 36$$

Figure 1. Arriving at an equation from abstracting the regularity in numerical guesses (Cuoco, 2018, p. 3)

This example involves what Cuoco refers to as "the dreaded algebra word problem," where he insists that we think of the *answer to the algebra problem as an equation* rather than a number – in a method that involves abstracting from numerals. The problem is as follows: *"Mary drives from Boston to Chicago, travels at an average rate of 60 MPH on the way down and 50 MPH on the way back. The total driving time takes 36 hours, how far is Boston from Chicago?"*

The method that Cuoco suggests builds upon students' ability to solve similar problems in middle school (note: they have already learned the relationship between speed, time, and distance) and is as follows: *Take a guess* – but the aim is not intended to get closer to the answer with each succeeding guess; rather it is to arrive at an equation, not a number. The idea is to carry out enough guesses so as to see the regularity of the calculations that allow for checking the guesses – in Cuoco's words: Develop "a generic 'guess checker' that is the desired equation". The processes of mathematical practice that are employed here, and which are ones that Cuoco declares he uses all the time in his own mathematical work, are:

1. Abstract regularity from repeated calculations, and
2. Use precise language (and algebraic symbolism) to give a generic and general description – the equation – for how you check your guesses. (Cuoco, 2018, p. 4)

The conclusion to be drawn from this example is that these two processes of mathematical practice fit well with the programming and thinking-like-a-programmer characteristics of computational thinking (Wing, 2006, 2014), and that students who are currently engaged in using digital technologies (e.g., laptops, robots) to code with visual (e.g., *Scratch*) or text-based languages are participating in mathematical practices. Nevertheless, other research (e.g., Bråting & Kilhamn, 2020) suggests that, while the representations used in programming languages may be similar to mathematical notations, the meanings of several concepts in the two domains differ. But that is a whole other story! In any case, digital technologies afford multiple varieties of mathematical activity that can offer experiences that involve coding but also those that do not.

Some Canadian research on the use of digital technologies to foster mathematical thinking

I take *mathematical thinking* to include the various processes that have been drawn upon by Wing and others to characterize aspects of computational thinking – but also more than this, for example, its conceptual aspects. While computational thinking is focused toward coding, mathematical thinking occurs within a host of activities that are not coding oriented, but which can clearly be engaged in within specifically-designed digital environments. However, the tricky thing about terms such as *computational thinking* and *mathematical thinking* is their overlap when referring to anything mathematical. Moreover, as Cuoco (2018, p. 2) has pointed out: "In real mathematical practice, it is rare that a piece of work employs only one aspect of mathematical thinking" – and, similarly, only one aspect of computational thinking. Despite the obvious intersection between the two terms, I find it helpful when discussing the use of digital technologies in mathematical activity to distinguish between *coding-related activity* and *non-coding-related activity.* In line with this distinction, I offer some examples that give a flavour of Canadian research that has focused on these two types of activity, both of which have successfully combined selected aspects of computational thinking and of mathematical thinking.

Digital Technologies in Coding-Related Mathematical Activity

***Scratch* coding on laptops.** My first example is drawn from the funded, multi-study research project of George Gadanidis and colleagues from across Canada, titled *Computational Thinking in Mathematics Education* – a project aimed at researching the use of computational thinking (via, e.g., digital tangibles such as circuits, programmable robots, and coding with *Scratch* on laptops) in mathematics education, from pre-school to undergraduate mathematics, and in mathematics teacher education (see ctmath.ca/about). In one of the publications from this project (Gadanidis et al., 2017), the initial activity engaged in by the Grade 1 students of a school in Ontario was the use of the block-based, visual programming language, *Scratch* (available at http://scratch.mit.edu), for exploring squares by drawing a set of squares rotated around a point (see Fig. 2; see also Gadanidis, 2015). One of the fundamental principles underpinning these study projects is connecting the digital technology work in classrooms to the math curriculum that teachers need to teach.

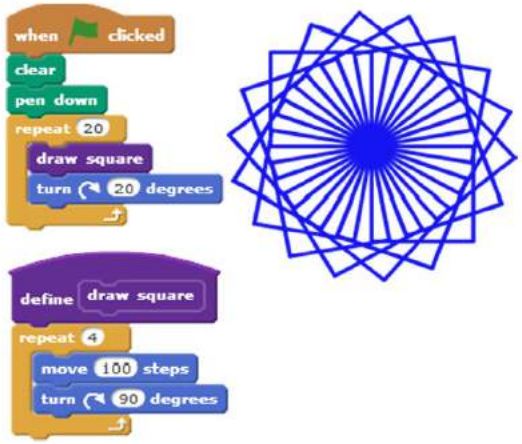

Figure 2: *Scratch* coding in Grade 1 (from Gadanidis et al., 2017, p. 81)

Figure 3. Programming a robot using loops (from Francis & Davis, 2018, p. 82)

Coding robots. Francis and Davis (2018) studied 9- and 10-year-olds' understanding of number, and the transition from additive to multiplicative thinking, in the context of learning to build and program Lego Mindstorms EV3 robots. The sequence of tasks focused on students' becoming aware of the architecture of robots, programming the robots to trace a triangle, square, pentagon, or hexagon; and building a robot that could find and douse a 'fire' in any of four rooms of a miniature model building. In one of the scenarios that Francis and Davis report on, a student learns how the number of sides and angles of a polygon connects to the number of repeats in a loop, which illustrates a developing shift from thinking additively in terms of a sequence of like actions to thinking multiplicatively in terms of a repetition of a single action (see Fig. 3). The authors argue that coding-related activity with digital technologies can co-amplify mathematics learning, as long as computer programming is seen as "something for" and is integrated into the existing curriculum with well-designed tasks, not as "something more" in a separate curriculum.

Digital Technologies in Non-Coding-Related Mathematical Activity

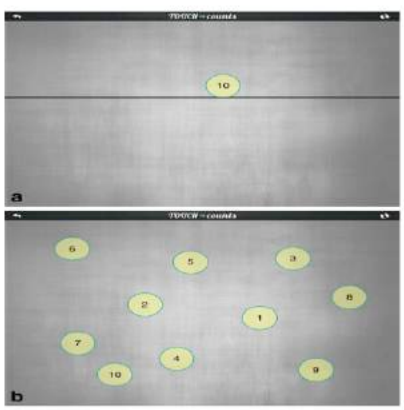

Figure 4. *TouchCounts* App: upper -- 10th tap; lower -- result of 10 single taps (Rodney, 2019, p. 169)

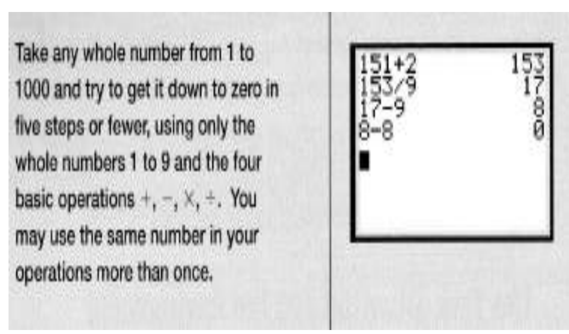

Figure 5. "Five Steps to Zero," with a starting number of 151 (adapted from Williams & Stephens, 1992)

***TouchCounts* – an iPad touchscreen App.** The *TouchCounts* application software, developed by Sinclair and Jackiw (2014), served as a window for the researcher Rodney (2019) to study how a 5-and-a-half-year-old, Auden, thought about number. Although Auden was able to say the number names initially, he seemed unaware that the written numeral '10' would appear right after '9' and that '10' also represented the number of taps made on the iPad screen (see Fig. 4). Auden's unsuccessful initial activity with the App revealed that his memorized number chanting needed the further support that *TouchCounts* could afford in order to reach a fuller understanding of counting and to begin to identify the relational aspect of numbers.

Calculators with multi-line screens. Calculators remain a staple in many mathematics classes. This resource, one with a multi-line screen, served as the digital tool underpinning a study that focused on the mathematical practice of seeking, using, and expressing structure in numbers and numerical operations (Kieran, 2018). The study (co-conducted with José Guzman†) involved classes of 12-year-old Mexican students on tasks adapted from the "Five Steps to Zero" problem (Williams & Stephens, 1992; see Fig. 5). Successfully tackling the designed tasks, and subject to the rules of the game, involved developing techniques for reformulating numbers (prime or composite) into other numbers in the same neighbourhood (not more than 9 away from the given number) that have divisors not larger than 9 so as to reach zero in five or fewer steps. Some of the most powerful structural explorations that occurred during the week of activity on the tasks involved the search for multiples of 9. For example, students became aware that "738 and 729 are two adjacent multiples of 9 and, when they are both divided by 9, the quotients are consecutive," and "in the 9-number interval from 735 to 743 inclusive, there is exactly one number divisible by 9." In trying to explain the often-surprising results produced by their digital tools, the students developed several mathematical insights that were new to them.

Carolyn Kieran's concluding remarks

My concluding remarks pick up on the interest shown by students in the use of digital technologies – be they coding-related or not. For example, Gadanidis et al. (2017) emphasize "learning experiences that offer the pleasure of mathematical surprise and insight" (p. 80). Similarly, Sinclair, Healy, and Noss (2015) speak of the "sense of delight" offered by digital technologies, but also of the need for a certain degree of "intellectual travel" (p. 2). In this latter regard, an early Logo study by Idit Harel (1990) is exemplary. Her 4th graders took up the challenge of designing and programming fractional representations that they thought would be helpful for younger children. This project led to significant gains in their understanding of both fractions and programming. As Harel points out: "the children's involvement in a rich, meaningful, and complex task, designing and programming a 'real' product for 'real' people, enhanced their understanding of Logo and their knowledge of how to use it" (p. 30). Clearly, embedding computational thinking into disciplinary contexts can be most productive and yields a strong lesson for policy-makers who advocate for stand-alone, coding programs in school.

The Use of DT in Mathematics Education: Experiences from Anglo-Mexican Collaborative Research And Implementation Programs in Mexico, by Teresa Rojano

Potential of the use of DT in the teaching and learning of algebra

There is abundant literature on research pertaining to the use of digital technologies for teaching and learning algebra, in which it can be observed a distinction between two types of tools, those developed expressly for this mathematical domain, such as *Computer Intensive Algebra*, *Cabri-Géomètre, Geometer Sketchpad*, *SimCalc*, and those that have been adapted for educational use, such as *Computer Algebra Systems* (CAS) and *Spreadsheets* (Sutherland & Rojano, 2014). Studies carried out with both types of programs have shown their strengths and limitations (Olive et al., 2010), and have highlighted the relevance of task or activity design in order to use them to create significant teaching environments (Donevska-Todorova et al, in press). A common denominator among many of these digital technologies, which is also one of its main potential strengths, is the connection between different representations, which allows for avenues to be created, where students can approach notions and learn novel and powerful algebraic methods, in an intuitive way (Zbeik & Heid, 2011).

Spreadsheets fall under the latter case, as the interconnection between the representation of numerical tables, the graphic representation and the use of algebra-like formulas, allows for the possibility of shifting between numerical and quasi-algebraic treatments, both for concepts like variable, unknown and function, and for word problem solving methods. Here I will specifically refer to my direct experience recreating intrinsic concepts and processes of algebraic thought, in this environment.

Spreadsheets and algebraic thinking. Outcomes from the *Anglo-Mexican Spreadsheets Algebra Project* (Sutherland & Rojano, 1993) carried out during the 90s, showed that students of different ages and school levels can work with algebraic ideas, taking a numerical approach, and using formulas whose syntax is similar to that of algebra. One of the studies was undertaken with two groups of students 10 to 11 years of age, one group in Mexico and the other in the UK, who worked with spreadsheet activities focused on the notions of function and inverse function, as well as on equivalent algebraic expressions and the solution of algebra word problems (Rojano & Sutherland, 1994). Figures 6 and 7 show examples of the type of worksheets used in the study.

The results of the pre-questionnaire applied to the students from both groups, before the experimental work, showed that the majority of the pupils did not think spontaneously in terms of a general object. Initially their mode of thought was on specific cases (for instance, on a particular line of a variation table) and the activities with spreadsheets helped them to move from focusing on particular cases to considering a general relation (see the worksheet in Figure 6). What is more, the sequence of activities on functional relations, the sequence on solving word problems helped the students to accept the idea of working with unknowns, to represent relations among data and the unknown of a problem, and to vary the numerical value of the unknown until they found the solution to the problem (see the worksheet in Figure 7). The experience with such activities enabled them to go from intuitive strategies for solving problems (such as trial and error, for example) to strategies in which they systematized their trials (trial and refinement) to finally encapsulate the relations between unknown and data in spreadsheet general formulas (Rojano & Sutherland, 1994; Sutherland & Rojano, 1993).

Function and Inverse Function

A) Copy the rule of the function that appears in the figure. Find the value of the function, when x = *30* ____________.

B) In B4, change the value of x to 100. Find the value of the function, for x = *115* ____________.

C) Make the x values vary in steps of 2. Copy down the function rule, when x is between *1* and *200*. Find the value of the function, when x = *150* ____________.

D) Try to vary the x values in steps of *0.5*.

	A	B	C
1			
2		x	Function of x
3			
4		1	6
5		2	7
6		3	8
7		4	9
8		5	10
9		6	11
10		7	12
11		8	13
12		9	14
13		10	15
14		11	16
15		12	17

Figure 6. Example of one of the worksheets provided for the teaching of function and inverse function.

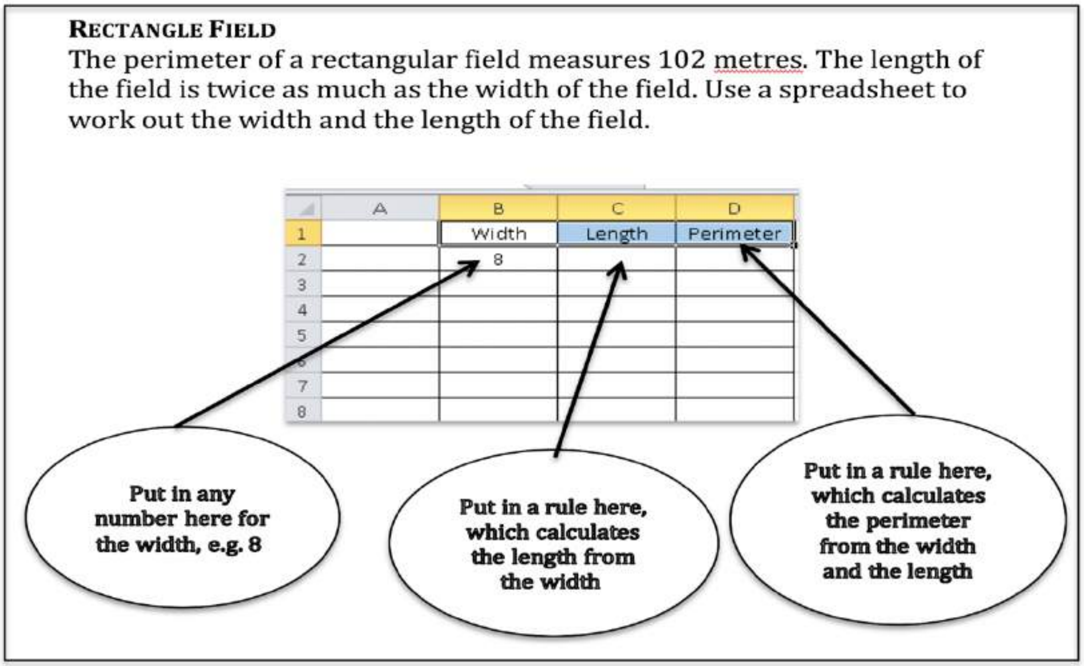

Figure 7. Rectangular field problem (Rojano & Sutherland, 1993).

In general, it is worth noting that despite the different experience in school mathematics that British and Mexican students could have had, given the differences in the mathematics teaching approaches in Mexico and the UK, there were no significant differences in the performance of the Mexican and British students, neither in the results of the pre-questionnaire nor during the work with the spreadsheets. The latter may be attributed to the connection that the students could have made between their own notions and intuitive strategies and the algebraic notions of function and unknowns, thanks to the combination of a numerical approach and the use of spreadsheet formulas, characteristic of the didactic design of the activities. In a second phase, the Anglo-Mexican project focused on working with 14 to 15 year-old students from both countries, students with a history of school failure in mathematics and who had already been introduced to the study of symbolic algebra. The results of the pre-questionnaire revealed that participating students were able to solve the problems using intuitive strategies and in some cases those strategies led them to the right solution. During the experimental work of this phase, activities with spreadsheets were used that were very similar to those of the study involving 10-11 year olds, in the end achieving the same effect –a connection between intuitive and non-formal approaches of the students, and notions and methods on the path toward algebraic thinking (Rojano & Sutherland, 1994).

Although spreadsheets were not developed for educational purposes, both the study undertaken with pre-algebraic students (10 -11 year-olds) and the study involving algebra resistant secondary school pupils show that use of that program accompanied by worksheets with an appropriate didactic design has great potential as a digital learning environment for exploring algebraic ideas and concepts.

The surprising work that participating students carried out in those studies did not prevent us from identifying the limitations of the environment that we used, which relate to what Zbiek et al. (2007) call 'mathematical fidelity'. In my interpretation, the distance between the syntax of Spreadsheet formulas and algebraic syntax may be a hallmark of weak mathematical fidelity; in the former case, formulas allow for representation and manipulation of generalizations, but they cannot be transformed with internal rules from that system of signs; while in the latter case, on the one hand, the analytical expressions of functions can be analyzed and transformed, under the rules of algebraic syntax so as to delve into the variation phenomena that they represent and, on the other, in advanced mathematics courses, those analytical representations can be treated as entities of more abstract mathematical structures. This continuity through the different educational levels is absent in the Spreadsheet environment. For its part, the spreadsheet method for solving word problems is mathematically and didactically pertinent to some families of problems, but it is far from the Cartesian method of solving problems, which is a general method and consists of getting the situation described in the text of the problem ‘put into an equation’. In summary, finding the didactic connection between the versions of notions and methods used in the digital technologies and the 'paper and pencil' versions used in school mathematics is still a significant challenge for teachers, trainers and curriculum designers.

In addition to understanding its potential and limitations, one should recognized that, together with the first dynamic geometry programs, the adaptation of Spreadsheets in mathematics education can be considered as part of the background to open source software Geogebra, which indisputable widespread use has more recently led to a large number of reports of practical experiences and research that use this package and that, among many other things, highlight the didactic feature of being able to create connections between algebra and Euclidean, Cartesian and Analytical geometries.

Research experiences: Windows of mathematics school culture

From our first forays into collaborative research on technology and algebraic thinking in the late 1980s, Rosamund Sutherland and I found differences between the educational systems of Mexico and the United Kingdom, some of which appeared specifically in the presentation of the topics of algebra in the curriculum, as well as in the diverse various educational material to be used in class. Both the differences and the commonalities permeated the design of the tasks used in the studies we undertook. However, observation and analysis of the ways in which each of the two groups solved the same task served as a window that allowed us to glimpse the distinctive features of the students' mathematical practices, where multiple representations of the same situation or phenomenon play a central role.

Spreadsheets as a mathematical modeling tool. More recent versions of spreadsheets offer a suitable environment for mathematical modeling tasks using (hot-linked) graphical, symbolic and numeric representations of phenomena of the physical world. The activities in this environment correspond to a parameterized version of the behavior of the modeled phenomena, and knowledge of advanced mathematics is not necessary in order to explore and build the models. In the Mexican-British project, *The role of spreadsheets within the school-based mathematical practices*[3], research was carried out with two groups (one in Mexico and another in the UK) of pupils of 16-18 years of age, in which it was observed how the differences of school culture experienced by each of the groups influences both their mathematics practices and their mathematics modeling activities using spreadsheets (Molyneux, Rojano, et al., 1999).

In the pilot phase, important differences were observed in the preferences of students for certain external representations of situations and phenomena. For example, when answering questions about the long-term behavior of the phenomenon, UK students showed a clear preference for graphic representation, while students in Mexico were inclined to use algebraic representation. However, even though work with Spreadsheets, in the experimental phase, did not significantly change those preferences, in the end the students recognized the value of having a varied repertoire of representations. In fact, one of the relevant results of this research was that participants developed the

[3] The Anglo-Mexican Project was developed as a collaboration between the Institute of Education of the University of London and the Department of Mathematics Education of the Centre for Research and Advanced Studies (Cinvestav) in Mexico, funded by the Spencer Foundation of Chicago, Ill, Grant No. B-1493.

ability to move smoothly from one representation to another and to realize the advantages afforded by some of them in answering certain types of questions posed about the behavior of the modeled phenomena (Rojano & García Campos, 2017).

Notwithstanding the fact that the main purpose of our research was not to study or compare school culture contexts, when analyzing the data collected in the two countries differences emerged that could not be explained, other than on account of what was emphasized or valued in the mathematics classroom, that is, it could be said that the differences originated in the school mathematics culture.

Research and Practice. The use of DT in the Mexican Educational System

In Mexico, as in many other countries, it has not been easy to bridge the results of research on the role of DT in the teaching of mathematics and incorporating their use in the educational system. We have gone through government programs ranging from the use of TV (in the 1960s) to broadcast live or pre-recorded classes, to the *Teaching Mathematics with Technology* (EMAT), *Enciclomedia* and the *New Model for the Telesecondary System.* The EMAT project was conceived specifically for the subject of mathematics in middle school, based on results from research in mathematics education. An international team of researchers designed a constructivist pedagogical model and student-centered activities, with an exploratory approach that encourages bottom-up practices, rather than traditional top-down practices. The tools used were Spreadsheets, Cabri-Géomètre, the TI-92 Algebraic Calculator and Logo, and a gradual implementation was planned starting in 1997, that would expand the use of different tools in different states of the country. Despite the fact that said implementation was not carried out as planned, some teachers who participated in the initial stages have continued to work for several years with EMAT activities, managing to integrate use of the tool repertoire into their own long projects (Trouche et al., 2013).

The experience from the EMAT project was used in the design and implementation of the New Model for Telesecundaria (Lower secondary system of rural areas without access to regular schools), which main feature is the articulation of printed, video and digital interactive resources, which are still in use. It is worth mentioning that the results of a study carried out by the Ministry of Education revealed that since the New Telesecundaria Model was launched, this system, compared to that of regular secondary schools, showed more sustained progress in terms of improved student scores on the yearly national mathematics and language exams (SEP, 2016).

Other national programs were suspended shortly after they were started, however, together with EMAT, Enciclomedia and the New Model for the Telesecundaria, the infrastructure and diverse experiences left by the programs throughout the country have been used by some teachers who have adapted the activities and the use of different tools to the curricular changes derived from the educational reforms. Whereas other teachers have limited the use of technology to displaying Powerpoint material, video material and YouTube, as it has been recently documented by Salinas, Sacristán and Trouche (2018). Thus, the use of DT to fundamentally transform mathematical practices at school and outside of school continues to be a great challenge in this country.

Reflections on the integration of digital technologies for mathematics education, by Ana Isabel Sacristán and María Trigueros

Taking into account the previous authors' reflections on the way digital technology culture has evolved in different countries' curricular approaches, teachers' professional development and students' participation and learning, we present some additional thoughts and discuss also observed contrasts and what happens in societies with different socioeconomic and cultural backgrounds. We also consider to what extent the potentialities of digital technologies have been harnessed to enhance mathematics learning and to engage students.

The previous authors have given us a panorama of the digital technologies that have been used in mathematics education. To summarize these, we refer to the USA's NCTM Position Statement which states that, in addition to content-neutral technologies such as tools for communication and collaboration and Web-based digital media, mathematics' content-specific technologies that can support students in exploring and identifying mathematical concepts and relationships, "include computer algebra systems; dynamic geometry environments; interactive applets; handheld computation, data collection, and analysis devices; and computer-based applications" (NCTM, 2015). To that we can add other expressive technologies such as computer programming environments, as well as eBooks.

On the potentials and historical evolution of digital technologies for mathematical learning

As Hoyles discussed above, it is clear that today's technology-based society needs to develop students' abilities not only in the use of technology, but as reasoners to exploit their use. The question is how digital technologies can be integrated and exploited in schools for achieving that and enhancing mathematical learning.

As many researchers in mathematics education, the panellists and ourselves believe (and a wide corpus of research has demonstrated) that digital technologies have the potential to change education, the teaching and learning of mathematics (e.g., by opening windows to mathematical knowledge, as Hoyles suggested), the curriculum, and also rethink mathematics as a field. Also, as Hoyles also mentions, using digital technologies in innovative ways could potentially widen access to learning.

Both Hoyles and Kieran cite Seymour Papert's vision of having students use computers as tools to think with and of how mathematical understandings can be changed by the use of digital technologies. Papert's (1980a, 1971/1980b) vision was for students using and programming computers for *doing* mathematics, rather than learning *about* mathematics; it is interesting that more recently, Chevallard (2015) also called for students to do mathematics, rather than visit them as if looking at a monument.

As explained by Hoyles above, programming allows users to interact with and visualize mathematics in new ways; it also can be used together with other technological tools so that it is possible to construct relations among concepts that so far have been considered as different, avoiding compartmentalization of knowledge.

In any case, an important potential of technology is for *empowering* students, be it through computer programming or through providing mediums for them to express themselves as well as mathematical ideas, and to build their own products (which is why Papert, 1991, called this *constructionism*) – in Hoyles' words, where students are producers, rather than consumers, of technology. And as Hoyles also emphasizes, Papert's vision had two central ideas: epistemological pluralism and microworlds. Hoyles discusses epistemological pluralism above, but we could add that when one has multiple means and representations to engage, interact with, and express mathematical ideas, those ideas become less abstract (Wilensky, 1991) and can more easily be appropriated. As discussed by Hoyles, computer programming, in particular, helps students to interact with mathematical objects in different ways, enabling students to visualize and reflect on the factors involved in different types of interactions.

The other important idea is that of microworlds, which Hoyles above described as a place where powerful mathematical ideas can be explored 'in safety'.

In the advent of the digital age, the Logo programming environment provided an ideal medium and microworld for students to engage in this epistemological pluralism and develop mathematical thinking, through computer programming and computational thinking. As touched upon by Hoyles and Kieran, following the publication of Papert's 1980's book *Mindstorms,* Logo had a widespread influence on the use of digital technologies in (mathematics) education. It was thus that Logo was implemented in countries around the world during the 1980s (including Mexico); and it was a crucial turning point in countries such as the UK and Canada, as well as the USA.

And yet, for a couple of decades starting in the 1990s, there was a generalized abandonment of Logo and Papert's vision, which happened in most countries. Kieran mentions a lack of funds for Logo projects losing ground in Canada, but the general reasons for that abandonment around the world are much more complex, and involve a confrontation with what Sacristán (2017) calls the 'inertia of school cultures'. Agalianos, Noss & Whitty (2001; see also Agalianos, Whitty & Noss, 2006) explain this complexity by pointing to how technologies and their use in the classroom are socially contextualized and often appropriated in ways unanticipated by their developers – something that is still true today:

> To work with Logo in the way its developers had envisaged meant that teachers should make a fundamental shift in their relationship with pupils….This, however, was a lesson not learnt for many teachers who felt uncomfortable […] Logo (...) automatically constituted a disruption of the classroom's traditional social organisation. [...]
> Logo's introduction into mainstream US and UK schools in 1980 marked the beginning of **a struggle to integrate new forms of teaching and learning into old educational structures**. (p. 486-487, our emphasis).

On his part, Ruthven (2008; see also Ruthven, 2014) points to the conjunction of several influential factors that are needed for successfully uptaking an innovation in school mathematics, and discusses how difficulties related to each of those factors acted against the continuation of the use of Logo in schools (and, we argue, in general, of more innovative visions, including Papert's). Those factors are:

- *Disciplinary congruence* with an influential contemporary trend in scholarly mathematics.
- *External currency* in wider mathematical practice beyond the school.
- *Adoptive facility* in terms of ease of incorporation into existing classroom practice.
- *Educational advantage* through perceived benefits of use considerably outweighing costs and concerns. (Ruthven, 2008)

In terms of adoptive facility, in the 1990s other developments, such as graphing calculators, CAS, Excel, dynamic geometry, etc. became popular and, as Kieran points out, seemed to offer teachers a more direct, less costly and perhaps easier to adopt, use of technology for mathematics teaching (and learning?). Many producers of such technologies also pushed newer software and hardware as solutions for the introduction of digital technologies in classrooms, but there is a difference between having a technology and using it for meaningful learning: we argue that, very often, schools became consumers of commercial developments, due to political and administrative decisions, failing to integrate those technologies in ways that would improve learning.

In any case, due to those complex reasons, in the 1990s and early 21st century the use of programming, and its relation to the development of mathematical thinking, was abandoned. During that period, in some countries, such as Mexico, there were efforts to return to something closer to Papert's vision through the development of the EMAT national project presented by Rojano, above. That project highlighted the importance of design and showed a way to use popular software, such as Excel, as in the example given by Rojano, in a way that could help in making the transition from arithmetic to algebra smoother for students. However, the EMAT project, though successfully tested, succumbed to political changes (Trouche et al., 2013), as we discuss further below; although, as Rojano points out, some teachers have continued using the EMAT technologies and materials in their classrooms until today, to promote their students' learning.

It is only in the last decade that interest and the recognition of the importance of computer programming, and computational thinking, has resurfaced in developed countries (but, as discussed below, not so much in other countries). Computational thinking, popularized by Wing (2006), and 'coding' have now become educational trends. How programming and computational thinking relate to mathematical thinking had already been discussed by Papert (1980a) and others in the 1980s and 90s. But Kieran above gives a profound discussion of their relationship. She agrees that computational thinking and mathematical thinking overlap, and engaging in programming and computational thinking implies participating in some mathematical practices. She argues, however, that mathematical thinking is more than what is achieved through coding and computational thinking but that other specifically-designed digital environments can also be helpful in promoting it.

Digital technologies in mathematics curricula across cultures

In any case, because of the attention that coding and programming have received in the past decade, many developed countries around the world have integrated these into school curricula.

In the UK, Hoyles mentions above how programming has come to the forefront since 2014; and Kieran describes experiences in Canada where programming has also been included in the curriculum. The question is how this integration affects the mathematics curriculum. Hoyles describes how she and her colleagues in the UK developed ScratchMaths in a research attempt to integrate the computer science curriculum with mathematics. It is interesting that ScratchMaths has been adopted in other countries outside the UK, such as Australia (Holmes et al, 2018), to use coding to teach mathematics.

In Canada, Kieran describes how programming activities have been introduced early in school but are used together with other technological tools to foster opportunities to construct relations among mathematical concepts.

However, in other countries, programming and computational thinking have not been included in the curriculum, nor sometimes even considered. In fact, there is thus a huge gap in the recognition of the possibilities that the use of digital tools offer to mathematics students and teachers and how the use of technology has evolved in countries and cultures around the world. The political strategies vary in different parts of the world to foster the inclusion of technologies in the mathematics classroom and rethink the curriculum.

In the USA, according to the NCTM: "All schools and mathematics programs should provide students and teachers with access to instructional technology—including classroom hardware, handheld and lab-based devices with mathematical software and applications, and Web-based resources—together with adequate training to ensure its effective use" (NCTM, 2015). This statement indicates a public recognition of the necessity of integrating a wide range of digital tools for teaching mathematics at schools, as well as the importance of teacher training for an effective and strategic use of those resources in the classroom. However, it is striking that in that statement, there is no mention of programming and computational thinking, which, in other countries have been considered more prominently as central for the development of mathematical thinking.

In Mexico, as discussed by Rojano, although at the turn of the century the potential of digital technologies in the teaching of mathematics was recognized when different public administrations, between 1998 to 2006, launched national projects directed at middle and elementary school children – EMAT, the New Telesecundaria Model, and Enciclomedia (Trigueros et al., 2006) –, this is no longer the case. Unfortunately, when the government changed, those projects disappeared. Since then, only the use of calculators and office applications have been recommended, but not included, in the following curricular changes. (Only some teachers, who found them useful to promote students' mathematical learning continue to use the EMAT or Enciclomedia tools in their teaching).

Thus, in many countries, and perhaps for different reasons, schools and curricula have remained attached to the use of software and/or calculators that are aids to the set curriculum and are considered useful in mathematical problem solving and in helping students understand specific concepts. In other words, they are added on to existing curricula and used to do the same as before but with the technology add-on, instead of for innovative ways of interacting with mathematics.

How the learning potential of digital technologies is considered varies from country to country, but also in different regions inside countries (as in the case of Canada, discussed by Kieran). In some places, digital technologies are embedded in school culture and specifically in mathematics classrooms, while in others, this is not the case. Julie et al. (2010) had already presented perspectives from different parts of the world illustrating the diversity of access and implementation of digital technologies. Some of these differences persist, perhaps due to diverse policies but also because of socioeconomic differences and factors (also, in terms of how technology is used and perceived in schools, gender may also be an issue). More problematic is the fact that many parts of the developing world still have issues of lack of access to technologies, rely on very old hardware, if any is available at all (see, for example, Sacristán et al., in press), and teachers have insufficient or no training in their use and pedagogical integration.

Thus, economic and social problems result in a widening gap, not only between different countries, but also inside these countries since development, technological opportunities and even the possibility to have access to technology are not the same. As technology continues to quickly develop, the gap between students who have opportunities to use technologies for mathematical thinking and learning, and those who don't, widens, and inequalities among different populations increase.

This gap in opportunities along the whole schooling process, has an important impact in terms of equity and on students' future. Jobs will need more and more mathematical and computing abilities so students who have not had the opportunity to develop them in depth will find it difficult to find jobs, thus the social context in different countries is increasingly divided.

On integrating new technologies and adapting to their changes

When integrating digital technologies for learning in schools, there is a recognition of the importance to provide, in Hoyles' words above, "a framework for the design and implementation of computational tools for learning mathematics." The issue of design is paramount. In summarizing Papert's constructionism, Hoyles mentioned the two central ideas of epistemological pluralism and microworlds. The design of the learning and exploratory universes that microworlds are, need include epistemological pluralism and take into account different components, as described by Hoyles and Noss (1987): the student component, the pedagogical component, the contextual component and the technical component. EMAT was a project that designed a model that took into account all of those aspects (Sacristán and Rojano, 2009). But careful design is only as good as what can be taken up by the educational system. As in the case of why Logo and Papert's vision declined in the 1990s, there is also the issue of the difference between an intended design and how it is implemented. In projects like EMAT or ScratchMaths, a very careful research-based design involving many researchers in mathematics education, as well as some teachers, was carried out aiming at maximizing the (mathematical) learning possibilities. These designs included, not only mathematical tasks, but also pedagogical models, associated teacher training models and possibilities for scaling up in order to expand to cover all, or a majority of, schools at a national level. However, when implemented, many of the initial intentions are lost. In the EMAT project, teacher training was never achieved as designed, and the scaling up was suspended due to government changes. In ScratchMaths there was also the problem of fidelity that Hoyles referred to above (see also Hoyles and Noss, 2019): how faithfully teachers (and schools) adopt, or are able to do so, the intended design. This necessarily leads to results that are generally less than expected.

Also, technology designers look for ways to offer educational systems, software with more capability in terms of interactivity and in terms of what they believe is needed in schools. Research on the use of technologies is constantly adapting to better understand how teachers and students use new resources in order to incorporate change in teachers' training programs and to look at how and if implementation in the classroom works in terms of students' mathematical learning.

Results often do not meet designers' expectations and real use of technology at school do not match with them either. On the one hand teacher adaptation to new technologies takes time and on the other hand when teachers find out that some tools work well in terms of their student' learning they are unwilling to try something new which may or may not work. These problems, together with the lack of access to tools and the fact that teacher training programs do not reach all teachers and the need to convince many teachers who don't want to take risks of finishing curriculum on time or don't see how technology matches with official curricula, need to be taken into account if the use of digital technology is considered important for the learning of mathematics.

When analyzing the efforts in the past couple of decades of educational systems around the world to integrate digital technologies in schools (not necessarily for mathematics), the words of Healy (2006), although she was referring to the case of Brazil, are valid in the cases of many countries: the attempts may tend to emphasize the computer as a catalyst for pedagogical change, but "they fail to acknowledge the epistemological and cognitive dimensions associated with such change or the complexity associated with the appropriation of tools into mathematical and teaching practices" (Healy, 2006, p. 213).

Moreover, technologies change fast. Keeping up with those changes is difficult for teachers, particularly if they have to catch up by themselves and only highly motivated teachers who have had good opportunities to evidence positive changes in their students are willing to do the needed efforts to develop teaching plans for their students to work with technology. There are also institutional constraints that limit teachers' possibilities to fully use the potential of digital technologies. Sacristán (2017) discussed in depth some of the challenges for teachers (and schools) to meaningfully integrate digital technologies into teaching practices for mathematical learning. Hoyles above also mentions the pressure of preparing students for national examinations, but there are others, such as the time needed to cover specific curricular topics, or the way a topic is presented in the textbook. All these factors need to be taken into account when designing teachers training courses.

Projects that involve collaboration between university researchers and teachers have proved to play an important role in the possible success of implementation in schools and in the promotion of students' learning, as was the case of the example given by Hoyles as well as the EMAT project mentioned by Rojano. Collaboration offers teachers opportunities to discuss strategies to use technology to transform their practice and to develop technological skills to make decisions to guide students' mathematical thinking. This type of approach, together with other teacher training initiatives, are indispensable for teachers to overcome their difficulties.

Unfortunately, teacher training opportunities are unequally supported in different cultures. While in some places programs are offered continuously to teachers, in others teachers are left to develop their own strategies to introduce technology in their classrooms; thus teachers are in many occasions discouraged and leave technologies aside.

Another phenomenon that has been observed is that in most places research projects to introduce technology to the classroom create lots of activity in terms of groups of researchers and teachers working together to creatively devise ways of using new technologies to teach mathematics. However, when those projects end, this activity stops. Regrettably, the lack of continuity and the gap between research and school policies, is true everywhere: Even in developed countries, as is the case with many European projects, when the funding finishes, interesting innovations abruptly end, and researchers move on to different proposals. This cycle repeats over and over, leaving aside the possibility to develop long term proposals which could make an imprint in the educational system.

Only a handful of creative and motivated teachers continue using the materials developed for research projects or previous government initiatives, adapting them to their teaching, and developing strategies to use those proposed technologies with their students.

Teachers and school systems have difficulty adapting to the perpetual changes in projects, as well as in the technologies available. It is therefore important to develop long-term professional development strategies and programs so that teachers can develop knowledge to be able to cope with and adapt to the technological changes in a productive way.

Concluding remarks

Digital technologies have played an increasingly important role in changing the mathematics classroom. Research has come a long way in terms of how to harness the possibilities that technologies can offer in terms of developing students' mathematical learning, although schools today are very different than when digital technologies first became available for education. But we should not forget that in this information age where information and communication technologies dominate, there are also other content-specific digital technologies, such as expressive and computer programming environments, that are important to develop mathematical thinking and practices. In particular, the tools that are generally used (including, more recently, communication ones during the COVID pandemic), haven't fully led to a meaningful integration of digital technologies *in* and *for* mathematics education. Thus, despite the exponential growth and influence of digital technologies in society, innovative visions (such as Papert's) haven't fully come to fruition.

Many projects, developed around the world, have shown that students enjoy using digital technologies in the mathematics classroom and there is evidence of the potential of these technologies to promote students' learning. But, although we have clear evidence that they can be used to foster students' mathematical competencies and learning, the expected results of their use in the classroom are still far from those desired. There are some important areas, such as evaluation and assessment (including self-assessment), and activities outside school, for which the potential of technologies has not been harnessed to support students' learning.

An important area that needs attention is the need to foster teacher training programs to overcome an existing tendency of teachers to improvise in the classroom, and for developing creative ways to use technology, including for assessment and self-assessment.

Also, inequalities in different school systems around the world, as discussed above, can be bridged through the development of long term projects that foster equality in the access of technologies, rich teacher training programs and activities for students to promote their learning autonomy. As it is now, differences among educational systems are widening but also differences in terms of opportunities within specific systems are far from what would be expected in terms of mathematical learning for all. When social and economic conditions are considered at this particular moment in time when the COVID pandemia has strongly affected all countries, life has become dependent on the use of technology in many aspects; but the difference between children that have access to technology and those who don't (or only have access to state-produced educational TV programs, as is the case in Mexico) is expected to increase the education gap.

It is time to consider more thoroughly how to harness the technologies that are already widely available to students in most countries: in particular, mobiles and smartphones. These accessible technologies have spread quickly around the world and are used by many students, but their rich educational potential has not received enough attention from researchers and from policy makers. These technologies are powerful and it is shown that more people have access to these than to computers. These tools need to be taken into account to develop interesting ways for their users to develop mathematical thinking inside and outside the classroom, with diverse uses and applications. New developments need creative ways of thinking and of using the growing potential of digital technologies in order to provide opportunities for all students to have access to computational and mathematical thinking, thus providing them with a potentially better future in the technology-based society. But this also implies transforming school cultures – not an easy task, as the story of Logo and other innovations shows.

Nevertheless, it is necessary to focus on how to use the technology potential to help reduce access inequalities, by looking for creative ways to help children around the world develop their mathematical thinking potential, and contributing to the creation of a better world for all. At the same time, we should also continue to devote time and effort to consider what has been dubbed "Papert's 10%" (from his call in his keynote speech at the ICMI 17 Study in Vietnam 2006): how mathematics and mathematical practices can change due to the availability and access to digital technologies.

References

Agalianos, A., Noss, R. & Whitty, G. (2001). Logo in Mainstream Schools: *The struggle over the soul of an educational innovation. British Journal of Sociology of Education,* 22(4), 479–500. https://doi.org/10.1080/01425690120094449

Agalianos, A., Whitty, G. & Noss, R. (2006). The social shaping of Logo. *Social Studies of Science.* https://doi.org/10.1177/0306312706053809

Aguirre, J. M., Mayfield-Ingram, K., & Martin, D. B. (2013). *The Impact of Identity in K-8 Mathematics: Rethinking Equity-based Practices*. Reston, VA: National Council of Teachers of Mathematics.

Baker, B., Cooley, L. & Trigueros, M. (2000). A calculus graphing schema. *Journal for Research in Mathematics Education*, 557–578.

Benton, L. Hoyles, C., Kalas, I & Noss, R. (2017) Bridging Primary Programming and Mathematics: preliminary findings of design research in England *Digital Experiences in Mathematics Education,* pp 1- 24

Bos, B. (2011). Professional development for elementary teachers using TPACK. *Contemporary Issues in Technology and Teacher Education, 11*(2). http://www.citejournal.org/vol11/iss2/mathematics/article1.cfm

Boylan, M., Demack, S., Wolstenholme, C., Reidy, J., & Reaney, S. (2018). ScratchMaths Evaluation Report and Executive Summary *Education Endowment Foundation*

Bråting, K., & Kilhamn, C (2020). Exploring the intersection of algebraic and computational thinking, *Mathematical Thinking and Learning*. DOI: 10.1080/10986065.2020.1779012

Brown, N., Sentance, S., Crick, T., & Humphreys, S. (2014). Restart: the resurgence of computer science in UK schools. *ACM Transactions on Computing Education*, 14.

Chevallard, Y. (2015). Teaching Mathematics in Tomorrow's Society: A Case for an Oncoming Counter Paradigm. In Cho . (Eds) *Proceedings of the 12th International Congress on Mathematical Education*, pp. 173- 187.. Springer, Cham.

Confrey, J. Hoyles. C. Jones, D, Kahn, K. Maloney, A. Nguyen, K., Noss, R. & Pratt, D. (2010). Designing for diversity through web-based layered learning: a prototype space travel games construction kit. In Hoyles. C & Lagrange J-B (eds) *Mathematics Education and Technology- Rethinking the terrain* (pp. 19- 46). Springer.

Crespo, S. (2003). Learning to pose mathematical problems: Exploring changes in preservice teachers' practices. *Educational Studies in Mathematics*, *52*(3), 243–270.

Cuoco, A. (2018). *Mathematical practice* (AMS Blogs on Teaching and Learning Mathematics). https://blogs.ams.org/matheducation/2018/08/17/mathematical-practices/

diSessa, A. A. (2018). Computational literacy and "the big picture" concerning computers in mathematics education. *Mathematical Thinking and Learning, 20*, 3-31.

Donevska-Todorova, A., Trgalová, J., Schreiber, C., Rojano, T., & Karnman, D. (in press). Quality of task-design in technology-enhanced resources for teaching and learning mathematics. In A. Clark-Wilson A. Donevska-Todorova, E. Faggiano, J. Trgalová & H.-G. Weigand (Eds.) *Mathematics Education in the Digital Age: Learning, Practice and Theory.* Routledge.

Drijvers, P., Hoyles, C., Kortenkamp, U., Noss, R., & Osborn, J. (2016, November 30). *Discussion Panel: Changing the way people think, move and feel mathematically: The contribution of digital technologies.* CARMA and AMSI Workshop: Tools and Mathematics - Instruments for learning, Newcastle, Australia. https://carma.newcastle.edu.au/meetings/tools/#panel

Francis, K., & Davis, B. (2018). Coding robots as a source of instantiations for arithmetic. *Digital experiences in mathematics education, 4*, 71-86.

Gadanidis, G. (2015). *Coding for young mathematicians.* https://researchideas.ca/coding/workshop-handouts/Gadanidis-math-coding-book-v8-uwo-sample.pdf

Gadanidis, G., Hughes, J. M., Minniti, L., & White, B. J. G. (2017). Computational thinking, grade 1 students and the binomial theorem. *Digital Experiences in Mathematics Education, 3*, 77-96.

Harel, I. (1990). Children as software designers: A constructionist approach for learning mathematics. *The Journal of Mathematical Behavior, 9*(1), 3-93.

Healy, L. (2006). A developing agenda for research into digital technologies and mathematics education: A view from Brazil. In C. Hoyles, J.-B. Lagrange, L. H. Son y N. Sinclair (Eds.), *Proceedings of the Seventeenth ICMI Study Conference "Technology Revisited" – Part 2: Contributions* (pp. 213-220). Hanoi Institute of Technology & DIDIREM - Université Paris 7.. https://www.mathunion.org/fileadmin/ICMI/files/Digital_Library/icmi-study-17/ICMI17proceedingsPart2.pdf

Herbst, P., & Chazan, D. (2006). Producing a viable story of geometry instruction: What kind of representation calls forth teachers' practical rationality? In S. Alatorre, J. L. Cortina, M. Sáiz, & A. Méndez. (Eds.), *Proceedings of the 28th North American Group for the Psychology of Mathematics Education Conference* (Vol. 2, pp. 213–220). Mérida, México: UPN.

Holmes, K., Prieto-Rodríguez, E., Hickmott, D., & Berger, N. (2018). Using coding to teach mathematics: Results of a pilot project. In *Integrated Education for the Real World: 5th International STEM in Education Conference: Post-Conference Proceedings* (pp. 152–158). Queensland University of Technology, Brisbane, Australia,. https://researchdirect.westernsydney.edu.au/islandora/object/uws%3A49792/

Hoyles, C. (2018) Transforming the mathematical practices of learners and teachers through digital technology *Research in Mathematics Education,* Vol 13

Hoyles, C., (1993) Microworlds/Schoolworlds: The transformation of an innovation. In Keitel, C., Ruthven, K. (eds) *Learning from Computers: Mathematics Education and Technology*. NATO ASI, Series F: Computer and Systems Sciences, 121, 1-17.

Hoyles, C., & Noss, R. (1987). Synthesizing mathematical conceptions and their formalization through the construction of a Logo‐based school mathematics curriculum. *International Journal of Mathematical Education in Science and Technology*, *18*, 581–595. https://doi.org/10.1080/0020739870180411

Hoyles, C. & Noss, R. (1992) (eds) *Learning Mathematics and Logo*. Cambridge MA: MIT Press.

Hoyles. C & Lagrange J-B (eds) (2010) *Mathematics Education and Technology- Rethinking the terrain.* Springer. https://doi.org/10.1007/s11528-018-0295-4

INTEF (n.d.). *La Escuela de Pensamiento Computacional*. Instituto Nacional de Tecnologías Educativas y de Formación del Profesorado (INTEF), Ministerio de Educación y Formación Profesional, Gobierno de España. https://intef.es/tecnologia-educativa/pensamiento-computacional/

Julie, C., Leung, A., Thanh, N. C., Posadas, L. S., Sacristán, A. I., & Semenov, A. (2010). Some Regional Developments in Access and Implementation of Digital Technologies and ICT. In C. Hoyles & J.-B. Lagrange (Eds.), *Mathematics Education and Technology - Rethinking the Terrain* (pp. 361–383). Springer. https://doi.org/10.1007/978-1-4419-0146-0_17

Kieran, C. (2018). Seeking, using, and expressing structure in numbers and numerical operations: A fundamental path to developing early algebraic thinking. In C. Kieran (Ed.), *Teaching and learning algebraic thinking with 5- to 12-year-olds: The global evolution of an emerging field of research and practice* (pp. 79-105). New York: Springer.

Manches, A., & Plowman, L. (2015). Computing education in children's early years: A call for debate. *British Journal of Educational Technology*.

Molyneux, S.; Rojano, T.; Sutherland, R. & Ursini, S. (1999) Mathematical Modeling: the Interaction of Culture and Practice. *Educational Studies in Mathematics*. Special Issue Teaching and Learning Mathematics in Context, P. Boero, (ed.) 39: 167-183.

Monaghan, J, Trouche, L & Borwein, J (2016). *Tools and Mathematics: Instruments for Learning.* Springer. https://doi.org/10.1007/978-3-319-02396-0

NCTM (2015, July). Strategic Use of Technology in Teaching and Learning Mathematics: A Position of the National Council of Teachers of Mathematics. https://www.nctm.org/uploadedFiles/Standards_and_Positions/Position_Statements/Strategic%20Use%20of%20Technology%20July%202015.pdf

Noss, R. & Hoyles, C. (1996) *Windows on Mathematical Meanings: Learning Cultures and Computers.* Dordrecht: Kluwer Academic Publishers.

Noss, R. & Hoyles, C. (2006), Exploring Mathematics through Construction and Collaboration. In R.K. Sawyer (ed), *Cambridge Handbook of the Learning Sciences.* pp. 389- 405

Noss, R. & Hoyles, C. (2017) Constructionism & Microworlds. In Duval, E, Sharples, M, & Sutherland, R (Eds) *Technology Enhanced Learning: Research Themes* (pp. 29-35)., Springer.

Noss, R., Hoyles, C., Saunders, P., Clark-Wilson, A., Benton, L., and Kalas, I., (2020 in press) "Making Constructionism Work at Scale: The Story of ScratchMaths": In Holbert, N., Berland, M., Kafai, Y. *Designing Constructionist Futures: The Art, Theory, and Practice of Learning Designs*, Cambridge, MA: MIT Press.

Olive, J. & Makar, K. with Hoyos, V., Kee Kor, L., Kosheleva, O., and Sträber, R. (2010). Mathematical knowledge and practices resulting from access to digital technologies. In C. Hoyles & J-B Lagrange (Eds.) *Mathematics Education and Technology- Rethinking the Terrain, The 17th ICMI Study*, (pp. 136-177). Springer.

Papert, S. (1980a). *Mindstorms: Children, computers, and powerful ideas*. New York: Basic Books.

Papert, S. (1980b). Teaching children to be mathematicians vs. Teaching about mathematics. In R. P. Taylor (Ed.), *The Computer in the School: Tutor, Tool, Tutee* (pp. 177–196). Teachers College Press, Teachers College, Columbia University. [Originally published in 1971].

Papert, S. (1986) Beyond Cognitive: The Other Face of Mathematics *Proceedings of the Tenth Conference of the International Group for the Psychology of Mathematics Education (PME 10), plenary lectures*, pp 51- 59, University of London, Institute of Education

Papert, S. (1991). Situating constructionism. In I. Harel & S. Papert (Eds.), *Constructionism* (pp. 1–18). http://www.papert.org/articles/SituatingConstructionism.html

Papert, S. (2006). *From the math wars to the new math*. Keynote speech presented at the Seventeenth ICMI Study Conference: Mathematics Education and digital technologies, rethinking the terrain. Hanoi, Vietnam.

Rasmussen, C., & Ellis, J. (2015). Calculus coordination at PhD-granting universities: More than just using the same syllabus, textbook, and final exam. In D. Bressoud, V. Mesa, & C. Rasmussen (Eds.). *Making the connection: Research and teaching in undergraduate mathematics education* (pp. 107–115). Washington, DC: The Mathematical Association of America.

Rich, P. J., Browning, S. F., Perkins, M., Shoop. T., Yoshikawa, E., & Belikov, O. M. (2019). Coding in K-8: International trends in teaching elementary/primary computing. *TechTrends, 63,* 311-329.

Rodney, S. (2019). “The other ten”: Order irrelevance and Auden’s sense of number. *Digital Experiences in Mathematics Education, 5*, 166-177.

Rojano, T. & García-Campos, M. (2017). Teaching mathematics with an intelligent support in natural language. Tertiary education students working with parameterized modeling activities. *Teaching mathematics and its applications* 36(1), 18-30 https://doi.org/10.1093/teamat/hrw009.

Rojano, T. & Sutherland, R. (1994). Towards an algebraic notion of function. The role of spreadsheets. In *Proceedings of the 16th Meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education* (PME-NA), Vol 1, pp. 278-284. USA: Louisiana State University.

Ruthven, K. (2008). Mathematical Technologies as a vehicle for intuition and experiment: A foundational theme of the international commission on mathematical instruction, and a continuing preoccupation. *International Journal for the History of Mathematics Education* 3(2), 91-102, 12. http://www.educ.cam.ac.uk/people/staff/ruthven/publications.html

Ruthven, K. (2014). Frameworks for analysing the expertise that underpins successful integration of digital technologies into everyday teaching practice. In *The Mathematics Teacher in the Digital Era* (pp. 373–393). Springer, Dordrecht. https://doi.org/10.1007/978-94-007-4638-1_16

Sacristán, A. I. (2017). Digital technologies in mathematics classrooms: Barriers, lessons and focus on teachers. In E. Galindo & J. Newton (Eds.), *Proceedings of the 39th annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education* (pp. 90–99). Hoosier Association of Mathematics Teacher Educators. http://www.pmena.org/pmenaproceedings/PMENA%2039%202017%20Proceedings.pdf

Sacristán, A. I., Rahaman, J., Srinivas, S. & Rojano, T. (in press). Technology integration for mathematics education in developing countries, with focus on India and Mexico. In A. Clark-Wilson, A. Donevska-Todorova, E. Faggiano, J. Trgalová & H.-G. Weigand, (Eds.). *Mathematics Education in the Digital Age: Learning Practice and Theory*. Abingdon, UK: Routledge.

Sacristán, A. I., & Rojano, T. (2009). The Mexican National Programs on Teaching Mathematics and Science with Technology: The Legacy of a Decade of Experiences of Transformation of School Practices and Interactions. In A. Tatnall & A. Jones (Eds.), *Education and Technology for a Better World* (pp. 207–215). Springer. https://doi.org/10.1007/978-3-642-03115-1_22

Salinas-Hernández, U., Sacristán, A. I., & Trouche, L. (2018). Technology integration into mathematics classrooms: case study of a high school teachers' use of GeoGebra. In V. Gitirana, T. Miyakawa, M. Rafalska, S. Soury-Lavergne, & L. Trouche (Eds.), *Proceedings / Actes Re(s)source 2018 International Conference* (pp 88–91). https://resources2018.sciencesconf.org/data/Proceedings_Re_s_source_2018.pdf

SEP (2004). *Programa Enciclomedia: Documento Base*. Subsecretaría de Educación Básica y Normal. México. Secretaría de Educación Pública.

SEP DGME (2013). *Resultados históricos nacionales 2006-2013, Enlace*.

Sinclair, N., Healy, L., & Noss, R. (2015). Editorial. *Digital Experiences in Mathematics Education, 1*, 1-6.

Sinclair, N., & Jackiw, N. (2014). *TouchCounts* [iPad application software].

Sutherland, R. & Rojano, T. (1993). A spreadsheet approach to solving algebra problems. *The Journal of Mathematical Behavior*, 12, 351-383.

Sutherland, R. & Rojano, T. (2014). Technology and Curricula in Mathematics Education. *Encyclopedia of Mathematics Education*, pp. 601-505. Springer. https://www.worldcat.org/title/encyclopedia-of-mathematics-education/oclc/885543690

Trigueros, M., M. Lozano, I. Sandoval, A. Lage, E. Jinich, H. García y E. Tovilla (2006), Developing resources for teaching and learning mathematics with digital technologies in Enciclomedia, a national project, in C. Hoyles, J.-B. Lagrange, L. H. Son y N. Sinclair (Eds.), *Proceedings of the Seventeenth ICMI Study Conference "Technology Revisited" – Part 2: Contributions* (pp. 556-563). Hanoi Institute of Technology & DIDIREM - Université Paris 7. https://www.mathunion.org/fileadmin/ICMI/files/Digital_Library/icmi-study-17/ICMI17proceedingsPart2.pdf

Trouche, L., Drijvers, P., Gueudet, G., & Sacristán, A. I. (2013). Technology-Driven Developments and Policy Implications for Mathematics Education. In M. A. (Ken) Clements, A. J. Bishop, C. Keitel, J. Kilpatrick, & F. K. S. Leung (Eds.), *Third International Handbook of Mathematics Education* (pp. 753–789). Springer New York. doi:10.1007/978-1-4614-4684-2_24

Turkle, S. & Papert, S. (1992). Epistemological Pluralism and the Revaluation of the Concrete *Journal of Mathematical Behavior,* Vol. 11, No.1, in March, 1992, pp. 3-33

Wilensky, U. (1991). Abstract Meditations on the Concrete and Concrete Implications for Mathematics Education. In I. Harel & S. Papert (Eds.), *Constructionism.* Ablex Publishing Corp. http://ccl.northwestern.edu/1997_prior/Wilensky_AbstractMeditations.pdf

Williams, D., & Stephens, M. (1992). Activity 1: Five steps to zero. In J. T. Fey (Ed.), *Calculators in mathematics education* (NCTM Yearbook, pp. 233-234). Reston, VA: NCTM.

Wing, J. M. (2006). Computational thinking. *Communications of the ACM, 49*(3), 33-35. doi:10.1145/1118178.1118215

Wing, J. M. (2014). *Computational thinking benefits society* (40th anniversary blog of Social Issues in Computing). Retrieved from http://socialissues.cs.toronto.edu/index.html%3Fp=279.html

Zbeik, R., Heid, C, Blume, G. & Dick, T. (2007). Research on technology in mathematics education: A perspective of constructs. In F.K. Lester (Ed.), *The Second Handbook of Research in Mathematics Education* (pp. 1169-1207). Charlotte, NC: Information Age Publishing and NCTM.

Zbeik, R.M. & Heid, K. (2011). Algebra and technology: Using technology to make sense of symbols and graphs and to reason about general cases. In T. P. Dick & K.F. Hollebrands (Eds.) *Focus in High School Mathematics: Technology to Support Reasoning and Sense Making* (pp. 20-27). NCTM