STUDENTS' REASONING ABOUT MULTIVARIATIONAL STRUCTURES

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Covariation and covariational reasoning are key themes in mathematics education research. Recently, these ideas have been expanded to include cases where more than two variables relate to each other, in what is termed multivariation. Building on the theoretical work that has identified different types of multivariation structures, this study explores students' reasoning about these structures. Our initial assumption that multivariational reasoning would be built on covariational reasoning appeared validated, and there were also several other aspects of reasoning employed in making sense of these structures. There were important similarities in reasoning about the different types of multivariation, as well as some nuances between them.

Keywords: Multivariation, Covariation, Student reasoning, Mental actions

Covariation and the cognitive activities involved in reasoning about it have become important themes in mathematics education research (e.g., Johnson, 2012; Moore, Paoletti, & Musgrave, 2013; Oehrtman, Carlson, & Thompson, 2008; Thompson, 1994; Thompson & Carlson, 2017). Yet, work on *co*-variational reasoning has essentially been limited to examining two variables changing in tandem with each other. Mathematical and scientific contexts often include more than two variables that are potentially related to each other. For example, the quantities pressure, volume, and temperature of a fix amount of gas inside of a flexible container are given by $PV = kT$, where P, V, and T could all be changing simultaneously. Mathematical functions of more than one variable, $z = f(x, y)$, also contain this feature. Note that we use "variable" in this paper to generically mean any potentially varying value, including values of real-world quantities as well as mathematical function inputs and outputs.

Recently, Jones (2018) used the term "multivariation" to theoretically describe situations where more than two variables relate to and change with one another. However, we do not yet have empirical data on the reasoning students might employ in making sense of these situations. This study was intended to explore, open-endedly, types of reasoning students might use when asked to think about multivariation structures. We went into this study with the assumption that multivariational reasoning would be related to covariational reasoning. Thus, our guiding research question was as follows: When analyzed through the lens of previous work on covariational reasoning, what reasoning mental actions are observed in students as they are asked to discuss different multivariational structures?

Background Research on Covariation

Because of our assumption that multivariational reasoning would be closely connected to covariational reasoning, we briefly review here some research work on covariation (see Carlson, Jacobs, Coe, Larsen, & Hsu, 2002; Castillo-Garsow, 2012; Johnson, 2015; Thompson & Carlson, 2017). The central theme to this work is that covariation consists of imagining "two quantities [i.e. variables] changing together" (Castillo-Garsow, 2012, p. 55) in which "they are changing simultaneously and interdependently" (Johnson, 2012, p. 315). The work of Carlson et al. (2002) provided the field with a framework of covariational reasoning mental actions, and then the more recent work of Thompson and Carlson (2017) heavily revised this into a new framework. For our purposes, we use the newer framework by Thompson and Carlson, though we draw on one major aspect of Carlson et al.'s original work. In particular, in Carlson et al.'s (2002) original framework,

In: Sacristán, A.I., Cortés-Zavala, J.C. & Ruiz-Arias, P.M. (Eds.). (2020). *Mathematics Education Across Cultures: Proceedings of the 42nd Meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education*, Mexico. Cinvestav / AMIUTEM / PME-NA. https:/doi.org/10.51272/pmena.42.2020

the first mental action of covariational reasoning was (1) coordinating one variable with changes in another. We believe this to essentially mean that students *recognize the dependence* between two variables, in that they perceive a change in one to correspond to a change in another (see also Oehrtman et al., 2008). This first mental action did not find its way into the revised framework by Thompson and Carlson, but it is important to our study because of the connections it has to some of our results.

Beyond this first mental action, we then used the revised framework (Thompson & Carlson, 2017) for the remaining mental actions. These subsequent mental actions are: (2) imagining related, but asynchronous changes in variables (*precoordination*), (3) imagining generic increases/decreases between the variables (*gross coordination*), (4) coordinating the variables' values (*coordination of values*), (5) coordinating changes in variables' values in "chunks" (*chunky continuous*) and (6) imagining changes in the variables' values happening smoothly (*smooth continuous*). These six mental actions were used in our study as baseline codes to categorize and organize student statements, as described in the methods section. We also extended this work by identifying new mental actions pertinent to *multi*variational reasoning.

Multivariation

Multivariation consists of situations where more than two variables are related to and possibly changing in conjunction with each other (Jones, 2018). Conceptual analysis has revealed different possible types of multivariation, which we recap in this section.

Independent Multivariation

Jones (2018) described *independent multivariation* as situations where certain variables can be held constant while others vary. For example, in $F = GmM/r^2$, one can change the distance (*r*) to produce a different amount of force (*F*), while keeping mass (m) constant. Multivariable functions, $z = f(x, y)$, typically also behave in this way. Yet, in independent multivariation, what is held constant and what can change can be switched. In $F = GmM/r^2$ one can keep the distance (*r*) the same to see how *F* and *m* might covary with each other. It is critical to note, though, that independent multivariation is more than simply the covariation of two variables while holding the others constant. Rather, one can imagine *multiple* variables changing at the same time. For example, in $F = GmM/r^2$, r and m could both be changing simultaneously, each impacting how F changes. In $z = f(x, y)$, one could trace a path in $\mathbb{R}^2$ in which both *x* and *y* change at the same time, with *z* changing as one traces along that path (see also Martinez-Planell, Trigueros-Gaisman, & McGee, 2015). Finally, another aspect of this multivariation is that it can include as many variables as desired, such as $z = f(x_1, x_2, \ldots, x_n)$ having *n*+1 variables.

Dependent Mulvariation

Next, Jones (2018) described *dependent multivariation* as situations in which it is *not* possible to hold some variables constant while changing others. A change in any one variable in this situation will produce simultaneous changes in *all* other variables. Some real-world contexts exhibit this behavior (Bucy, Thompson, & Mountcastle, 2007; Roundy et al., 2015), such as $PV = kT$. If the gas inside the flexible container is heated up, the increase in temperature (*T*) would cause simultaneous changes in both the internal pressure (*P*) and volume (*V*). As another example, in free-market economics, if one changes the price of a commodity, both demand and supply will react simultaneously. It may not be realistic or even possible to hold "demand" constant to observe only changes in supply. Similarly, for parametric functions in mathematics, $\langle x(t), y(t) \rangle$, if one changes *t*, then *x* and *y* both change simultaneously.

Nested Multivariation

Third, Jones (2018) described *nested multivariation* as situations where the variables are related in a function composition structure. In the structure $z(y(x))$, as one imagines a change in x, there is a corresponding change in y. That change in y then automatically corresponds to a change in z. It may be necessary sometimes to *perceive* the intermediary variable if it is not explicitly labelled, such as $y = sin^2(x)$ consisting of the quantities x, sin(x), and y. As x changes, sin(x) changes, which in turn makes y change. Of course, it is possible to conceptualize two-variable covariation between x and y directly in this example. However, a complete understanding of their relationship would require interpreting the intermediary sin(x) value (see also Breidenbach, Dubinsky, Hawks, & Nichols, 1992). Otherwise, for instance, if x decreases into negatives, the values of y might not be accurately tracked. Real-world quantities can also have this nested structure. For example, in the theory of relativity, as an object's velocity changes, the object's mass changes, given by $= m_o/\sqrt{1 - v^2/c^2}$. As the velocity (v) changes, the ratio between it and the speed of light (v/c) changes, which in turn changes the Lorenz factor $1/\sqrt{1 - (ratio)^2}$, which in turn changes the mass (m) (see also Jones, 2015).

Study Methods

To document students' multivariational reasoning, we recruited 10 undergraduate students to participate in interviews, referred to as Students A–J. Students E, G, and J were female and the others male. Because our study was exploratory in terms of the types of reasoning students might use, we decided to recruit students who were more advanced in their mathematical studies, to better ensure that they had had exposure to and experience with multivariational contexts. Similar to how Carlson et al. (2002) recruited second-semester calculus students to investigate covariational reasoning, we recruited students in multivariable calculus (from two different classes) to investigate multivariational reasoning. In the interview, the students were given two contexts for each type of multivariation (Table 1). These contexts were chosen for their connection to the conceptual analysis that helped define multivariation (Jones, 2018). For each context, the students were allowed to clarify the context first, and then were asked, "What does this equation/formula mean? What does it say about the variables in it?" The students open-endedly discussed the context, but were also asked several scripted questions, including how the variables related to each other, how changes in one variable impacted the others, whether multiple variables could change at the same time or whether variables could remain unchanged, and what impact increases or decreases in certain variables might imply.

Table 1. Contexts Given to the Students in the Interviews

Multivariation	Context 1	Context 2
Independent	Let $z = x^2 - y^2$ be a function of two variables. [The function's graph was also given to the student.]	The formula $F = \frac{GMm}{r^2}$ relates gravitational force (F) with mass (m) and distance (r). M (Earth's mass) and G are constants.
Dependent	For a certain amount of gas in a flexible balloon, $PV = kT$ relates pressure (P), volume (V), and temperature (T).	The price (p) of a specific book is related to the number of books people want to buy (d for demand) and number of books the publisher is willing to print (s for supply).
Nested	Let $y = \sin(x)$ and $z = y^2$. In other words, $z = \sin^2(x)$.	$m = \frac{m_o}{\sqrt{1-(v^2/c^2)}}$ relates an object's mass (m) to its velocity (v). Note that m_o is the "resting mass" and c is the speed of light.

Our analysis was based on our assumption that multivariational reasoning would be related to covariational reasoning, though also extended beyond it as well. Thus, our initial analysis consisted of using the covariation mental actions described previously as starting codes. We marked any place in the data where a student exhibited reasoning behaviors similar to one of the covariational reasoning mental actions. While we did, we also used open coding to mark any reasoning instances that did not align with one of the covariational reasoning mental actions. After doing so, we examined these "other" reasoning instances in order to identify themes among them. This led to the emergence of new codes that were not a part of the covariational reasoning mental actions. Once we had our final set of codes, we recoded the entire data set again, using our completed coding scheme. Then, within each of independent, dependent, and nested multivariation contexts, we compared the reasoning used across the 10 students. We looked for trends in how the students tended to reason about each multivariation type. We also compared the reasoning used in one type of multivariation with reasoning used in another to identify if certain kinds of reasoning were distinctive to one type of multivariation or common across them.

Results

Our first main result was that the students did, in fact, employ much covariational reasoning within these multivariation contexts, supplying evidence for our assumption that multivariational reasoning is rooted in covariational reasoning. We observed all of the mental actions from Thompson and Carlson (2017) in our students. In fact, as seen subsequently, imagining only two variables at a time was a common and useful action that students did. Space constraints do not permit a full treatment of how each aspect of covariational reasoning was observed, and we instead focus the remainder of our results on reasoning mental actions specific to multivariation.

Students' Independent Multivariational Reasoning

In the independent multivariation contexts, all 10 of our students engaged in reasoning that was related to Carlson et al.'s (2002) first *recognize dependence* mental action. Yet, a slightly different aspect of that reasoning in these contexts was a similar mental action we call *recognize independence*. In this, the students decided which variable was to be treated as constant and which were to vary. For example, one of the first things Student B said when shown $z = x^2 - y^2$ was, "x and y are variables, independent variables. Which basically means as one changes the other doesn't necessarily have to change." In $F = GmM/r^2$, Student C stated, "As I am getting farther from the earth with a bowling ball, I'm not changing the mass of the bowling ball." Recognizing independence then permitted the students to use another new mental action that we call *decompose into isolated covariations*. For instance, when discussing $z = x^2 - y^2$, Student D early on stated, "Whether x is increasing or decreasing… it is going to be increasing the z either way." Then a few statements later, Student D described, "Let's just pretend that the x^2 doesn't exist and we're only playing with the y^2… We see the parabola for y, which is negative, it starts at the origin and then curves down in both directions." Here, Student D simplified the context to two variables at a time in order to understand the covariational relationships between x-z and y-z. After using this mental action to reduce the multivariation to covariation, covariational reasoning mental actions were then used to analyze that relationship between those two particular variables.

In conjunction with recognizing independence and decomposing into covariations, a third new mental action we observed was that students could *switch constants/variables*, in which they shifted their conceptualization of which variables were held as constant and which were allowed to vary. For example, a little after the previous excerpt, Student B explained, "If we follow this path $x = y$, our z stays constant." Similarly, Student C later stated, "Increasing the mass, getting a pebble compared to a rock and having them the same distance from the earth, I have increased the mass but the distance is still the same." These students demonstrated they were able to reason from different perspectives within the same context in terms of what changes or stays the same.

The students were also able to perform mental actions regarding the variables all changing at the same time. We call one such mental action *imagining simultaneous changes in inputs*. In this mental action, two variables were considered the "inputs" and their changes were imagined as linked together *before* then coordinating this with the variable considered to be the output. To illustrate, in discussing $z = x^2 - y^2$, Student H explained, "I could move simultaneously in both an x direction and a y direction. That's going to determine how my z direction is changing… If I'm changing my x and my y at the same time, then z can potentially change as well."

Building on imagining simultaneous changes, another mental action for independent multivariation was what we call *coordinate these multiple simultaneous changes*. For example, in the force context, after Student G had first decomposing into isolated covariations and then subsequently imagined simultaneous changes, she explained, "Say the mass is increasing and the distance is also, r is decreasing, then the force would definitely be increasing." This mental action consists of aligning the results of the isolated covariations together into an overall image of all the variables' changes. Student G also considered the possibility of m and r both increasing or both decreasing. She explained, "If the mass and the distance are both increasing or both decreasing, then it gets a little bit iffy. It depends on which one has a greater impact. [Pause] If m is increasing, at a rate that's greater than the rate at which r^2, the distance squared, is increasing, then the force will still increase." Here we can see Student G comparing the covariations between m and F and r and F. She decided that if m changes by more than r^2, then the positive covariation between m and F will overcompensate for the negative covariation between r and F.

In Student G's explanation, we also see another important mental action. Here, she did a mental action close to what Thompson and Carlson (2017) call *coordinating values*, but she did so without ever using specific numeric values. Thus, from our study, we decided that covariational and multivariational reasoning research would benefit from separating out what we call *qualitative amounts of change* versus *numeric amounts of change*. In other words, Student G was able to *qualitatively* image that the increase in F due to a large increase in m would be larger than the decrease in F due to a smaller increase in r^2. She could have done this by using specific numeric values, but her coordination at the qualitative level was productive for what she wanted to accomplish. We see this mental action as applying to both covariation and multivariation.

Lastly, another new mental action we saw was students attempting to *articulate the type of relationship* present between two or more variables. It appeared helpful for students to determine well-known relationships present between the variables. For example, for $z = x^2 - y^2$ Student I explained, "You maybe pick some value of x and keep it there and then you just basically have $z = c - y^2$. So it's just an upside parabola." Visualizing a parabola helped him think of how z and y would covary with each other. Students used other well-known relationships to assist imagining the situation, such as thinking of F and m as *proportional*, and F and r^2 as *inversely proportional*.

Students' Dependent Multivariational Reasoning

Recall it is not possible to hold some variables constant in dependent multivariation. Thus, an important mental action students used was, again, *recognize independence/dependence*. Yet, the way this mental action was carried out varied from student to student. For instance, in the $PV = kT$ context, Student H explained, "So, if my temperature were increasing… I can think of both my pressure and my volume increasing. The balloon is getting bigger and the pressure inside it is increasing." Here, Student H envisions a dependent relationship between all three variables simultaneously. However, when Student A was asked if T could change so that only P changes, without V changing, he explained, "If you just keep, I mean, is this according to the equation?

According to the equation, then I would say yeah [V can be kept constant], because if V is constant and P increases, then T would increase." Student A did accept that all three could be changing simultaneously, but also observed that in the mathematical equation, one can leave one as constant. These examples illustrate two things. First, whether a context is independent or dependent multivariation consists, at least in part, in how the person *conceptualizes* it. Regardless of how things behave in the "real world," if a student perceives that it is possible to hold some variables constant, then that is the type of situation they cognitively work with. Second, whether the students chose to conceptualize it as independent or dependent multivariation seemed connected with whether they believed they should operate mentally in a "math" world of the symbolic equations, or the way quantities behave in the "real" world.

When students determined that they were in a dependent multivariation situation, they also often decomposed into isolated covariations. But, then, the mental action *coordinate simultaneous changes* became important. For example, after decomposing $PV = kT$ into covariations, Student D stated, "If the balloon is being heated up, then its volume will greatly increase and its pressure will increase a little bit, depending on the capacity of the balloon to contract." He put the individual relationships together into a coherent whole. However, an important difference in dependent multivariation is that the simultaneous nature of the changes is *required*, where it is not required in independent multivariation. This excerpt again shows *qualitative amounts of change*, because Student D imagined relative changes without using exact numeric values. As another example, Student J stated, "If the increasing amount of T is greater than either [the change in] P or V, that means both would be increasing." She realized that if P and V both increase, they could not each increase relatively as much as T does by itself. Of course, students did engage in *quantitative amounts of change*, too, such as Student E examining what possibilities he could get for changes in P and V if temperature changed from "6 to 10."

Like for independent multivariation, in these contexts the students also spent time trying to *articulate the type of relationship* between the variables. For example, several students discussed "proportionality" between V and T, or "inverse proportionality" between P and V. In the supply and demand context, students also used ideas of proportionality and inverse proportionality. Some students tried to create a rough symbolic formula to relate them, which took the forms of $p \propto \frac{d}{s}$ (Student A), $p = k\frac{d}{s}$ (Student B), $p = d - s$ (Student F), and $d = \frac{p}{s}$ (Student H). Note that different relationships can be seen depending on which quantity was seen as changing first. For example, if price is seen to increase first, that might signal a decrease in demand. Alternatively, if demand increases first, that might signal an increase in price. Other students drew graphs of p versus d and p versus s that matched these equations. *Articulating these relationships* seemed to help students organize their thinking about relative changes between the variables. In particular, Student J drew the familiar supply and demand curves (Figure 1) and used them to help her organize her thinking of relative changes. She first imagined that where the decreasing demand curve intersected the increasing supply curve defined the price. An increase in demand was represented as a shift upward in the demand curve, which resulted in an intersection point at a higher supply and higher price (left image). Similarly, an increase in price (right image) resulted in a point lower along the demand curve but higher along the supply curve. This kind of reasoning seems to suggest another possible mental action, *identifying order of effect between variables*.

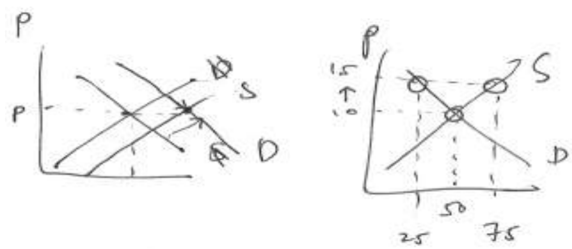

Figure 1: An example of *articulating relationships* helping students organize their thinking

Students' Nested Multivariational Reasoning

For this type of multivariation, as with the other types of multivariation, the students employed mental actions regarding *recognizing* relationships. In this case, the students attempted to *recognize a chain of influence* from one variable to the next. For example, with $z = sin^2(x)$, Student E imagined an intermediary quantity in the chain. She explained, "So, you put the *x* in, the *x* gets sine'd, and then that sine gets squared. So, it goes in like step one, it's turned into a sine, and then step two that sine is squared." Student E recognized that there was (a) the initial *x* value, (b) the sine of that *x* value, and (c) the square of that sine value. In the mass-velocity context, Student E similarly explained, "The velocity is never going to be more than the speed of light. So, this [points to v^2/c^2] is always going to be less than one, which means this [point to $1 - v^2/c^2$] will always be positive. But the more velocity increases, the closer this [points to v^2/c^2] is going to get to one, which means the smaller the denominator [e.g., $\sqrt{1 - v^2/c^2}$] is going to get, which means the larger the mass is going to be. So, the larger velocity gets, the larger mass gets." In this, Student E conceptualized explicitly the quantities (a) v, (b) v^2/c^2, (c) $\sqrt{1 - v^2/c^2}$, and (d) m. Several other students gave similar descriptions of these two contexts.

Once the chain of influence had been recognized, students again often used *decompose into isolated covariations*. To illustrate, as Student B thought about $z = sin^2(x)$ he described, "As x changes, we're going to end up with *y* having this oscillating pattern… between positive and negative one repeatedly. So, as you increase *x*, your *y* is going to be jumping between 1 and –1. If you decrease *x*, same thing… If *y* is positive, as we increase *y*, *z* will go up. If *y* is negative and we decrease *y*, *z* will also go up." Student B first examined *x* and *y* in isolation and then *y* and *z* in isolation. As before, these isolated covariations then needed to be *coordinated* into an overall image of the nested multivariation. Additionally, students also employed coordination of increases and decreases, including both *qualitative amounts* and *numeric amounts*.

Once students had completed mental actions of recognizing a chain of influence, decomposing into isolated covariations, and coordinating increase/decrease, these seemed to help students understand the direct relationship between the "initial input" variable (i.e. *x* and *v*) and the "final output" variable (i.e. *z* and *m*). They could take their new knowledge about the context and begin to work with direct covariation between the initial input and the final output. They did not necessarily need to work with the intermediary variables anymore. For example, after working through the nested reasoning, Student A summarized the velocity-mass context as follows, "As it's [*v*] changing, so if this gets bigger, then *m* would get bigger as well… So, if this [*v*] increases, *m* would increase and if this [*v*] decreases, *m* would decrease." Thus, one part of understanding nested multivariation structures might be to eventually conceptualize the direct two-variable covariation between the two most salient variables of interest.

As a last note, some students also attempted to circumvent the need for nested reasoning for $z = sin^2(x)$ by instead using visual reasoning on the graph of $\sin^2(x)$. They first took the graph of $\sin(x)$ and attempted to reason what the square of that graph looked like. Once they had a graph (whether correct or incorrect), the students then used that graph to vary *x* and report directly on its impact on the values of *z*.

Discussion

First, we observed that covariational reasoning was, in fact, an important part of these 10 students' reasoning. In all three multivariation types, students often *decomposed* the context to two-variable covariations at a time to organize their thinking. However, we note that this decomposition into covariations has connection to what Johnson (2015) termed simultaneous yet independent variation. In her work, she explained that students sometimes covaried two variables with *time* and then tried to coordinate the variables only through the intermediary of time. Johnson concluded that a full comprehension requires students to not need the intermediary of time, and to imagine the variables changing directly in relation to each other. Likewise in multivariation, it may be important for students to push past decomposition into covariations to imagining a single coherent image of the whole multivariational structure. Some of our students were able to compile the individual covariations to create a holistic image of the multivariation.

Next, our study extends covariational work by elaborating on *coordination of values* from the Thompson and Carlson (2017) framework. We suggest it be split into two types of coordination: *qualitative amounts of change* versus *numeric amounts of change*. Our students productively described "large" or "small" changes qualitatively to reason about a context. This mental action certainly goes beyond gross coordination. We even hypothesize that it may even be more complex than simply inserting numeric values into a formula and comparing resulting values (i.e. coordination of values), because it requires one to imagine relative sizes in changing values and coordinate them without the aid of specific numeric values. We perhaps even see chunky continuous coordination (Castillo-Garsow, 2012; Thompson & Carlson, 2017) as just adding intervals of a fixed size to *qualitative amounts of change*. Thus, we wonder if *qualitative change* may be between what is currently described as coordination of values and chunky continuous coordination. Of course, additional work would be required to examine if that is the case.

Third, another key idea from our study is that it requires cognitive work to *recognize dependence* and *independence* among the variables in multivariation. Students spent time imagining what variables could be held constant, which varied, which depended on which, and whether that dependence could be altered. In a recent paper, Kuster and Jones (2019) similarly noticed the importance of "recognize" in students using variational reasoning while discussing differential equations. They claimed that it may have been an oversight to drop "recognize" from the original covariational framework (Carlson et al., 2002) in the new framework (Thompson & Carlson, 2017). Our data concurs that it may be important to keep mental actions of "recognize" in variational reasoning frameworks, because of how important it is for more complicated variational structures. This suggests that in moving our students from covariation to multivariation, it may be useful to spend time engaging students in *recognizing* activities. It is possible we do not help students see, for example, the difference between independent and dependent contexts (see Bucy et al., 2007; Roundy et al., 2015).

Finally, we saw that there was much similarity in the types of reasoning across the different multivariation contexts. The good news is that it might not be necessary for students to learn about each type of multivariation separate from the others. By learning to reason about one type, they may simultaneously be developing reasoning abilities that transfers to other types. However, by being explicit about the different types, we as instructors might help students gain an appreciation for the nuances that exist between each type, enabling stronger reasoning.

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