MIDDLE SCHOOL STUDENTS' DEVELOPMENT OF AN UNDERSTANDING OF THE CONCEPT OF FUNCTION

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Middle school students (n=144) worked with an applet specially designed to introduce the concept of function without using algebraic representations. The purpose of the study was to examine whether the applet would help students to understand function as a relationship between a set of inputs and a set of outputs and to begin to develop a definition of function based on that relationship. Results indicate that, by focusing on consistency of the outputs the students, at a rate of approximately 80%, are able to distinguish functions from non-functions. Also, students showed some promise in recognising constant functions as functions, a known area of common misconceptions.

Keywords: Middle School Education, Technology, Representations and Visualization

Introduction

The concept of function is considered to be one of the most important underlying and unifying concepts of mathematics (e.g., Leinhardt, Zaslavsky, & Stein, 1990; Thompson & Carlson, 2017). Students have experiences with functions, or function behaviour, from the very earliest grades usually through pattern exploration. Study of functions continues up to and through high school with a formal treatment of functions as arbitrary mappings between sets. Indeed, in the Common Core State Standards for Mathematics function is given its own domain in grades 9-12 (Common Core State Standards Initiative, 2010).

Much of the lack of depth of knowledge of the concept can be attributed to the privileging of algebraic representations (function as algebraic rule) or graphical representations (function as graph that passes the vertical line test) and a consequent lack of focus on the general relationship (see e.g. Best & Bikner-Ahasbahs, 2012; Breidenbach et al., 1992; Carlson, 1998; Thompson, 1994). What might a group of students who have never encountered the concept of function learn by encountering it in a novel representation? Can they learn to think of a function as a relationship between inputs and outputs with some rules about the outputs rather than something that is defined by an algebraic rule? These are the questions that guided the current study.

Related Literature

Prior to secondary school, opportunities for study of functions are limited in scope (Best & Bikner-Ahasbahs, 2012; Carlson & Oehrtman, 2005; Vinner & Dreyfus, 1989) and focus mainly on pattern recognition and study of covarying quantities, most often related to an underlying linear structure (Blanton et al., 2015; Stephens et al. 2017, Ellis, 2011). For example, in Blanton et al. (2015) 6th grade students are given the tasks "People and Ears: The relationship between the number of people and the total number of ears on the people (assuming each person has two ears)" (p.520) to study the function type $y = x + x$ and "Age Difference: If Janice is 2 years younger than Keisha, the relationship between Keisha's age and Janice's age (Carraher et al., 2006)." (p. 521) to study the function type $y = x + 2$. In other words, the functional relationships typically encountered in elementary and middle school years are designed to prepare the (mathematical) ground for studying linear relationships ($y = mx$, $y = x + b$, $y = mx + b$) i.e. the privileging of algebraic representations begins early in the study of functions. Leinhardt et al. (1990), in a meta-study of research on function, and Mesa (2004), in a study of 24 middle grades textbooks from 15 countries, note the

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difficulty for students in apprehending the modern, abstract definition of function depending, as it does, on the mapping of one set of elements to another emphasising the difference between function and relation (many-to-one acceptable, one-to-many not acceptable); whereas, the work on function in early grades builds on the intuitive notion of a 1-1 correspondence and the historical development of function rested on covarying quantities.

Even in secondary school functions are typically introduced as very limited classes such as linear and quadratic, with attendant graphs and tables, with the result that students regularly consider functions to be mathematics objects solely defined by an algebraic formula (e.g., Best & Bikner-Ahasbahs, 2012; Breidenbach et al., 1992; Carlson, 1998) and have difficulty identifying constant functions as functions (Bakar & Tall, 1991; Carlson, 1998; Rasmussen, 2000). Instruction and curricular materials often emphasize procedures and algebraic manipulations when studying functions and research shows that students then have difficulty in understanding different representations and different contexts for functions (Carlson & Oehrtman, 2005; Cooney et al., 2010). At the heart of many student difficulties is a shallow understanding of the definition (Ayalon et al., 2017; Panaoura, et al., 2017). Students who have an algebraic view of function and who use procedural techniques to identify functions and non-functions struggle to comprehend a general mapping between sets (Carlson, 1998; Thompson, 1994).

Exposure to, and facility with, various representations of functions, i.e "flexible use of functions . . . within and between all kinds of representations and also between different functions" (Best & Bikner-Ahasbahs, 2012, p. 877), has been shown to be a critical component of a rich understanding of function (Best & Bikner-Ahasbahs, 2012; Dubinsky & Wilson, 2013; Martinez-Plandi & Tigueros Galsman, 2012). Furthermore, researchers have found promising results when using novel contexts and non-standard representations of functions such as dynagraphs, arrow diagrams, and directed graphs (Dubinsky & Wilson, 2013; Sinclair, Healy & Sales, 2009). The purpose of this study is to examine the effect of a specially designed applet on middle school students' ability to develop an understanding of the concept of function.

Methods

Context

Previous research (Meagher et al., 2019) has shown the promise of a vending machine representation as a "cognitive root" (Tall, McGowen, & DeMarois, 2000) for the study of functions. Thus, we designed an applet, *Introduction to Function,* (https://tinyurl.com/y2dramsb) as a mechanism for learners who have never encountered the concept of mathematical function and, therefore, do not associate the concept with any particular representation, to learn the basic elements of function. The goal was for the students to learn that a function is a relationship between of a set inputs that are matched with a set of outputs in a consistent and, therefore, predictable manner.

The *Introduction to Function* task is a GeoGebra book that consists of seven pages and has an accompanying worksheet. On the first two pages are two vending machines each of which consists of four buttons (Red Cola, Diet Blue, Silver Mist, and Green Dew). When a button is clicked it produces none, one, or more than one of the four different colored cans (red, blue, silver, and green), which may or may not correspond to the color of the button pressed (see Figure 1). The students are told that the first machine on each page is an example of something called a function, and the other is not a function, with their task being to identify what is the difference between the behaviour of the machines that makes one a function and the other not.

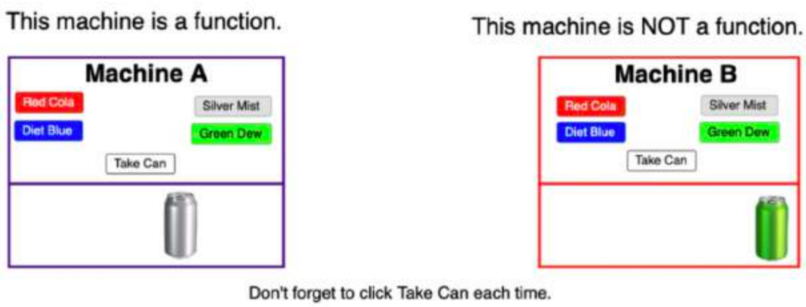


Figure 1: Screenshot of Introduction to Function

The machines on the first two pages work as follows:

This One is a Function		This One is Not a Function	
A	Red – Red Blue – Blue Silver – Silver Green – Green	B	Red – Red Blue – Blue Silver – Random Green – Green
This One is a Function		This One is Not a Function	
C	Red – Blue Blue – Red Silver – Silver Green – Green	D	Red – Red Blue – Random Silver – Silver Green – Green

Figure 2: Machines A – D

Note that Machines B and D are not functions because one of the buttons, when clicked, will produce a random can (i.e. not always the same result). Note also that in Machine C the colour of the output can does not correspond to the input button pressed, but that the non-matching can is consistently produced. After the first two pages there was a whole group discussion in which students discussed the first two pages, with the goal of consolidating their ideas.

The next four pages of the GeoGebra book consist of pairs of machines with the students being told that one of each pair is a function In each case there is a random element in the non-function. The machines work as follows:

	Which One is a Function?			
Page 3	E	Red Cola – red Diet Blue – blue Silver Mist – silver Green Dew – random color	F	Red Cola – silver Diet Blue – green Silver Mist – red Green Dew – blue
	Which One is a Function?			
Page 4	G	Red Cola – random color Diet Blue – random color Silver Mist – random color Green Dew – random color	H	Red Cola– blue Diet Blue – silver Silver Mist – green Green Dew – red
	Which One is a Function?			
Page 5	I	Red Cola – 2 silver cans Diet Blue – green Silver Mist – red Green Dew – blue	J	Red Cola – red Diet Blue – blue & random color Silver Mist – silver Green Dew – green
	Which One is a Function?			
Page 6	K	Red Cola – pair of random color Diet Blue – blue Silver Mist – silver Green Dew – green	L	Red Cola – green Diet Blue – green Silver Mist – green Green Dew – green

Figure 3: Machines E - L

On the worksheet, students are asked to note whether each machine is a function or not a function and how they know. After they complete these pages students are given the prompt: "Using the terms 'input' and 'output' write a definition for function based on your exploration of the machines."

Participants

The *Introduction to Function* applet was used in fifteen seventh grade classrooms. These classrooms were across two different states (one Northeastern state and one Southeastern state) and five different teachers for a total of 144 students who engaged with the task. These students engaged with the applet towards the end of their seventh grade year and had not yet learned about the definition of function or function notation.

Data collection and analysis

Students worked in pairs (N = 72) to engage with the applet on a laptop that screen captured their work. Data collected were their worksheets, which include their definitions, screen recordings, and audio recordings. For this study our analysis focused on the students' worksheets. All data was coded by three researchers. Any disagreements were discussed until any discrepancies were resolved.

For the definitions we coded for use of the terms input/output, attention to output, and focus (Author et al., 2019). In terms of input/output, each definition was read for use of those terms in the definition for example, "M49_M62: No matter what input the output is the same" and "M117_M118: A function is when you get the same output." In terms of focus, each definition was coded regarding whether the definition indicated a function was a relationship (or mapping), an object, or neither. We referred to this set of codes as focus, as they indicated how the students "saw" function. If the definition indicated that the function relates to the input and output then the definition was coded as a relationship. For example, "VM_M91_M96 The word function may mean when you input something, even though you may not get what you asked for, you will only get one type of it." The code "object" was used when the definition referred to a function as something, such as the button, or the machine.

Finally, definitions were coded according to whether or not they attended to output. In order for a definition to be coded as attending to output, the definition needed to refer to an output having a pattern, or being the same or consistent. For example, "VM_M54_M59: Function is when you put in the input and the output will never change / will always be the same."

Analysis of the student worksheets proceeded along two dimensions: classification of whether the pairs of students correctly identified the machines E through L as functions and the students' justifications for their classifications. For the pairs of machines E&F, G&H, I&J, K&L, since students were told one was a function and one was not, it was possible to simply count the classification. Of course, the percentages should mirror each other i.e. the number of "corrects" for machine E should match the number of "incorrects" for machine F.

The students' written justifications for their machine classifications were open coded using a constant comparative method to look for themes (Creswell, 2014). The final codes for students' justifications are shown in Figure 4. Justification codes were not mutually exclusive, as a justification could have been coded based on inconsistency as well as using the context of the vending machines.

Code	Description
Justification based on inconsistency of output	Students' justifications use phrases indicating an attention to the inconsistency of the outputs. Examples include: "different colors", "random", "it changes".
Justification based on consistency of output	Students' justifications use phrases indicating an attention to the consistency of the outputs. Examples include: "consistent", "constant", "pattern".
Justification uses the context of the machine	Students' justifications describe the relationships between inputs and outputs using the vending machine context. Examples include: "because it always gives the wrong drink", "it gives random colored cans"
Justification unclear	There is not enough detail in the students' written response to classify their justification.

Figure 4: Justification codes

Results

Identification of the Machines.

The first element of analysis was to tally whether the participants were able to correctly identify which of the machines E-L are functions. Recall that participants worked through machine pairs A&B and C&D being told that A is a function and B is not a function and that C is a function and D is not a function, and that the concept established was that the machine should behave consistently even if the colour of the output can does not match the colour of the button pressed. Students classification of the machines is shown in Table 1.

Machines	Non-function reason	% Correct
E & F	Machine E: Green Dew has random output	81.3
G & H	Machine G: all outputs are random	95.8
I & J	Machine J: Diet Blue output is Blue & random	86.1
K & L	Machine K: Red Cola output is 2 random cans	80.7

Figure 5: Participants' correct identification of functions

At a first level of analysis this shows that, broadly speaking, the pairs of students were able to correctly identify which machines were functions. The percentage of correctly identified functions for the first four pairs of machines was at least 80% and ranged from 80.7% to 95.8%.

It is interesting to note that for the pairs E&F, I&J and K&L the correct percentage is very similar (between 80% and 86%). The exception is the machine G&H pairing which has a much higher percentage of students identifying it correctly. This can be explained as follows: the primary identifying factor for a machine not being a function was the random behaviour of one of the buttons. However, one has to press a button often enough to be able to identify the behaviour as random. In the case of Machine G, all four buttons give random output and, therefore, the threshold to identify random behaviour is lower. Furthermore, Machine G comes first and, therefore, students can very quickly identify Machine G as not a function and not concern themselves too much with Machine H.

Looking more closely at the incorrect answers for the first four pairs of machines we see that it is often the same pairs of students getting incorrect answers. 10 of the 14 (71.4%) pairs of students who made a misidentification of the E&F pair misidentified at least one other machine, with 5 pairs misidentifying all of the first four sets of machines except the G&H pairing. Furthermore, of the 22 pairs of students that misidentified at least one machine, only seven of the 22 (31.8%) had their first wrong answer after the first pair of machines E&F and six of those seven misidentified just one of the pairs E&F, G&H, I&J and K&L.

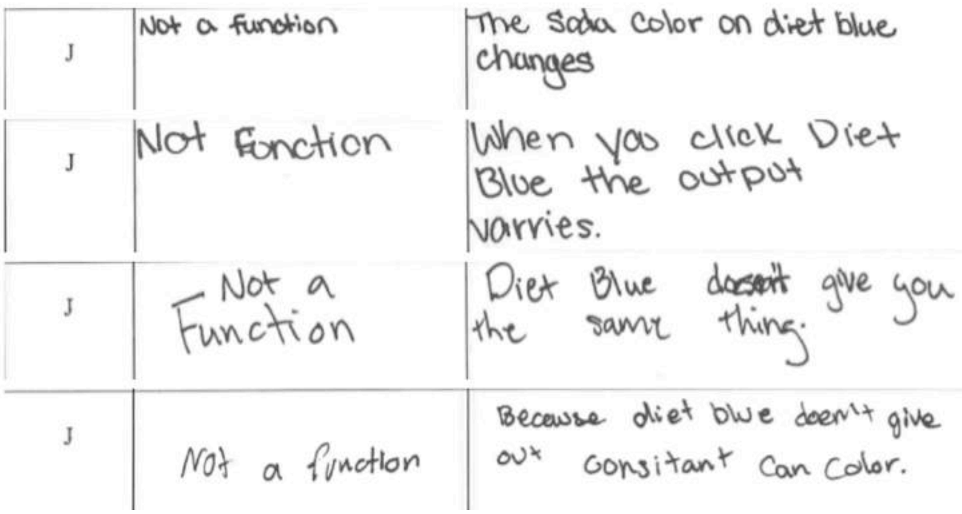
The result for Machine L with 80.0% of participants identifying it as a function is a potentially significant result since researchers have shown that students exhibit difficulties identifying constant functions as functions (e.g. Carlson, 1998; Rasmussen, 2000). However, it may be that many students identified Machine K (output from Red Soda is two random cans) as not a function and concluded that Machine L must be a function.

Characterizing Students' Justification of Functions and non-Functions.

To better understand the ways in which students were making sense of the machines, we analyzed their justification for whether or not each machine was a function or non-function (see Figure 5). Those that were determined to be functions were justified based on consistency of the input/output relationship and those determined to be non-functions were described as such based on the inconsistency of this relationship. One notable exception to this is the 11 students that used the language of inconsistency to justify their choices for Machine F (Red Cola → silver, Diet Blue → green, Silver Mist → red, Green Dew → blue). All 11 of the students that described this as inconsistent, also determined the Machine was not a function. We see that these students could not overcome the cognitive dissonance of a machine giving them a different colour output can from the input button pressed, even if it did so consistently. For example, one student (M90) described Machine E (R→r, B→b, S→s, G→ random) as "more consistent" than Machine F (R→s, B→g, S→r, G→b) which "randomizes things." The very next pair of Machines in the applet had a similar design (Machine H: R→b, B→s, S→g, G→ r), and only one student determined this to be a non-function using the reasoning of inconsistency. This suggests that the students refined their meaning for such a justification to be aligned with situations in which a single output results in different outputs. Examples of students' justifications based on inconsistency are shown in Figure 7 below.

Machine	Justification based on consistency	Justification based on inconsistency	Justification unclear
E	5	54	13
F	45	11	17
G	2	68	2
H	62	1	8
I	54	6	12
J	2	59	9
K	4	54	11
L	58	1	10
M	38	2	15
N	34	1	14

Figure 5: Characterizations of students' justifications for each machine

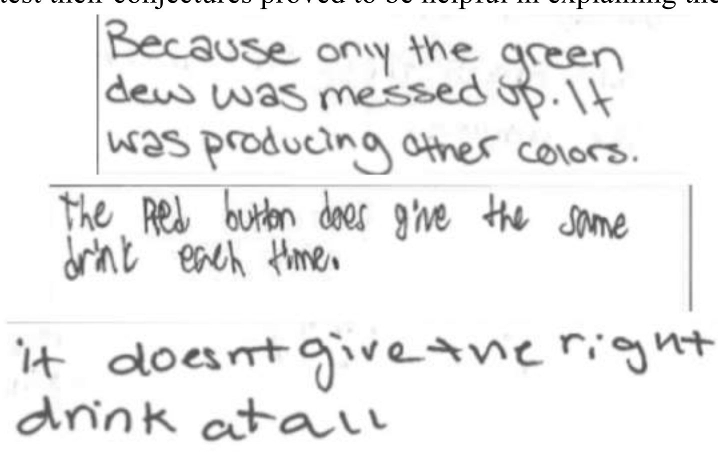


J	Not a function	The soda color on diet blue changes
J	Not Function	When you click Diet Blue the output varries.
J	Not a Function	Diet Blue doesn't give you the same thing.
J	Not a function	Because diet blue doesn't give out consitant can color.

Figure 7: Examples of justifications based on attention to inconsistency of outputs

As is evident in the Machine F example above, the students' justifications provide insight to their misidentification of both functions and non-functions. For example, looking at the 13 pairs of students that misidentified Machine K (R→random pair) as a function it is evident that they either did not test the machine enough to see the random outputs that occurred when clicking Red Cola (e.g., "every color is functional, red produces 2 greens"), or they decided that since the rest of the buttons were consistent it was "close enough". For example, one pair wrote "mostly consistent" and another wrote "3 of the 4 function correctly." Furthermore, the inability to accept machines giving a different output from the button pressed, even if it does so consistently, persisted for a number of pairs. For example, Pair M17 & M20 said of machine J (R→r, B→b & random, S→s, G→g "The Blue one gives two but the others work."

It is notable that 80% of the student pairs used the language of the machine context in their justifications (see Figure 8 for examples). This suggests that having a realistic context in which to both think about and test their conjectures proved to be helpful in explaining their thinking.



Because only the green dew was messed up. It was producing other colors.

The Red button does give the same drink each time.

it doesnt give the right drink atall

Figure 8: Examples of justifications that use the context of a vending machine

Definitions

One of the 72 pairs of students did not complete a definition on their worksheet. The remaining 71 definitions were coded using the codebook. In terms of the use of input/output 62 out of 71 (87.3%) definitions used the word input and 65 out of 73 (89.0%) definitions used the term output. Of course, the participants were asked to use these terms and, therefore, the result is not entirely surprising. Nevertheless, the result is promising in terms of establishing sets of inputs and outputs as a central aspect of the definition of function.

Perhaps the most interesting aspect of the activity was to examine the extent to which the participants would pay due attention to the outputs from the machines. Analysis of the definitions shows that 45/71 (61.6%) of the participants did pay attention to the output with definitions such as "When you input something, the output always will stay the same." However, 14/71 (19.7%) of participants, while paying attention to the output made an incorrect statement such as "Your input is your output and does not change."

In terms of focus, none of the participants described a relationship between inputs and outputs explicitly as a mapping between sets, and most definitions (43/71 (60.6%)) were coded as "neither object or relationship." A large number of participants' definitions (27/71 (38.0%)) were coded as "object" since they made explicit reference to the vending machine or the buttons of the machine. For example, "Whenever you input into the vending machine, you know the output which makes it reliable."

Conclusion

The purpose of this study was to explore whether seventh grade students who had not encountered the term function could use a specially designed applet to develop an understanding of a function as a relationship between inputs and outputs with some restrictions on the outputs. The non-standard representation of the *Introduction to Function* applet served to introduce the concept of function without algebraic representations. With the focus on the consistency, or otherwise, of the outputs the participants were able to correctly distinguish between functions and non-functions at least 80% of time. Some limitations of the study may be that the results were overdetermined by the discussion after the first two pairs of machines and that the participants might be seen to be simply playing a pattern recognition "game" with the rule "random bad, not random good." Therefore, more study would be needed to establish if the basic concept learned here transfers effectively to further study of function. However, even within this study, more than 60% used some appropriate language to describe the nature of the output in their definitions of function. In addition, contrary to a well-known misconception, participants may be able to recognise a constant function as a function.

Acknowledgments

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