UNIT TRANSFORMATION GRAPHS: A CASE STUDY

Anderson Norton
Virginia Tech
norton3@vt.edu

Catherine Ulrich
Virginia Tech
ulrich@vt.edu

Sarah Kerrigan
Virginia Tech
stk123@vt.edu

We introduce a methodology for diagramming the ways students use sequences of mental actions to solve mathematical tasks. We studied 12 pre-service teachers as they solved a set of fractions tasks, ranked by cognitive demand. We present the unit transformation graphs for one of those pre-service teachers, to illustrate how she experienced and met cognitive demand across the fractions tasks. Specifically, the graphs illustrate how sequencing mental actions places demands on working memory and how units coordination structures can offload some of that demand.

Keywords: Cognition; Learning Theory; Number Concepts and Operations; Problem Solving.

Prior mathematics education research from a Piagetian perspective has identified students' construction and transformation of units as central to their development of number, extended to rational numbers and even algebraic reasoning as generalized arithmetic (Hackenberg, 2013; Steffe & Cobb, 2012; Steffe & Olive, 2010). The present study was motivated by a desire to explicitly identify mental actions that undergird the construction and transformation of units in the context of fractions. For example, students might construct 1/7 as a unit, which has a one-to-seven relationship with a whole unit. This relationship might be established by the mental action of partitioning a continuous whole into seven equal parts. Conversely, the relationship could be reversed by iterating a 1/7 part seven times to reproduce the whole. Thus, partitioning and iterating constitute reversible and composable mental actions that can be used to construct a 1/7 unit and transform it back into the whole unit (Wilkins & Norton, 2011).

The purpose of this paper is to introduce a methodology for modeling students' mathematics by explicitly identifying the sequences of actions and unit structures they use to solve mathematical tasks. Unit transformation graphs (UTGs) account for the constraints of working memory in sequencing actions, as well as the power of unit structures in offloading demands on working memory. We share the case study of a pre-service teacher (PST) with relatively high working memory and the ability to assimilate and operate on two-levels of units. Findings (and the UTGs themselves) illustrate how students might use their units coordinating structures to chunk sequences of actions into single units, thus reducing the cognitive demands of mathematical tasks.

Theoretical Framework

Mathematics educators have begun to explicitly account for students' actions in building models of their mathematical reasoning. Recent examples include activity-effect relations (Tzur & Simon, 2004) and the Learning Through Activity framework (Simon Placa, Avitzur, & Kara, 2018). Similar to those models, UTGs explicitly identify the mental actions students use to construct and transform units, but have two distinguishing features. First, they account for the role of working memory in sequencing actions. Second, they explicitly account for the role of unit coordinating structures in reducing demands on working memory.

Mental Actions for Constructing and Transforming Units

Following Piaget (e.g., Beth & Piaget, 1966), we characterize mathematical actions (operations) as mental actions that are potentially reversible and composable. We are particularly concerned with operations students use to construct and transform units. For example, a student might construct a unit by isolating a collection of items, treating them as identical, and taking them as a whole—a

In: Sacristán, A.I., Cortés-Zavala, J.C. & Ruiz-Arias, P.M. (Eds.). (2020). *Mathematics Education Across Cultures: Proceedings of the 42nd Meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education*, Mexico. Cinvestav / AMIUTEM / PME-NA. https:/doi.org/10.51272/pmena.42.2020

mental action called unitizing (Steffe, 1991). Students might also unitize a continuous span of attention, whether it be time, length, area, or volume. Once constructed, units can be transformed into other units: students might iterate a unit, making copies of it and integrating the copies within a new composite unit (a unit composed of units); or they might partition a unit into equal parts, forming smaller units.

Steffe (1992) originally defined a units coordination as a distribution of the units within one composite units across the units of another composite unit. For example, in determining the value of 7 times 4, a student might distribute seven 1s across the four 1s that comprise 4, making a sequence of four units of seven units of 1. This definition orients our thinking about how units might be transformed into other kinds of units, but we include additional transformations as units coordinations. The aforementioned operations of unitizing, partitioning, iterating, and distributing are all potentially reversible and composable, and all can be used to transform units into other units. In addition, disembedding enables a student to remove a unit, or collection of units, from a composite unit, without destroying the composite unit (Steffe, 1992). The student maintains the composite unit while considering some of its parts as units separate from that composite unit.

Sequencing Operations in Working Memory

Students might need to perform a long sequence of operations to solve a mathematical task. In our framework, the cognitive demand of the task would increase with the length of this sequence. This perspective aligns with Pascual-Leone's (1970) characterization of working memory as a mental-attentional operator (the M operator). "Working memory involves the process of holding information in an active state and manipulating it until a goal is reached" (Agostino, Johnson, & Pascual-Leone, 2010, p. 62). It is a limited resource used to implement mathematical problem solving strategies (Bull & Lee, 2014; Swanson & Beebe-Frankenberger, 2004) and one that predicts children's mathematical achievement (Blankenship et al, 2018; De Smedt et al, 2009).

In the context of mathematical problem solving, Pascual-Leone (1970) characterized this limited capacity (M-capacity) as "the number of separate schemes (i.e., separate chunks of information) on which the subject can operate simultaneously using his mental structures" (Pascual-Leone, 1970, p. 302). Commensurate with other measures of working memory, Pascual-Leone (1970) found that adults can typically hold in mind 5-7 schemes at once. In numerical contexts, such as solving fractions tasks, schemas might refer to the operations students use to construct and transform units. A student might hold in mind a sequence of seven such operations, but fractions tasks may involve multiple levels of units (e.g., the whole, unit fractions, measures of a unit fraction) with many transformations between them. A student might offload some of that demand on working memory through the use of figurative material, such as drawings or notations, or by assimilating some of the units and unit transformations into existing cognitive structures: units coordinating structures.

Units Coordinating Structures

In the absence of structures for assimilating multiple levels of units, each unit or unit transformation (e.g., partitioning a whole into n equal parts) would place separate demands on working memory. However, those units and the operations that transform them can be organized within units coordinating structures (Boyce & Norton, 2016; Hackenberg, 2007; Ulrich, 2016). The rectangle on the right side of Figure 1 represents a two-level structure for coordinating units—one that would organize the previously-described one-to-seven relationship between a whole unit and the unit fraction, 1/7. Note that the structure contains two units and a pair of reversible mental operations between them: the whole can be transformed into seven equal parts by via the operation of partitioning; and this mental action can be reversed by iterating one of those parts seven times, reproducing the whole. This unit coordinating structure acts as a single unit that can be used to assimilate two units and an action (in either direction), thus reducing cognitive demand from three to one.

Figure 1: Units coordinating structures

The mathematical power of units coordinating structures is well documented (Boyce & Norton, 2019; Hackenberg, 2007; Tillema, 2013). For example, Hackenberg and Tillema (2009) demonstrated that students who can assimilate three levels of units are able to reason through fractions multiplication problems in ways that other students cannot, including students who can assimilate two levels of units. This power can be explained, at least in part, by reduced demands on working memory. When units are assimilated into existing structures, working memory is freed to focus on ever more complex tasks. This structuring and offloading also explains the sense in which mathematics builds upon itself.

Methods

The data collected and analyzed for this paper is part of a larger project investigating the cognitive development of mathematics, behaviorally and neurologically. This paper reports on results from video recorded behavioral data.

Data Collection

Participants consisted of PSTs at a large university in the mid-Atlantic United States. All participants were enrolled in one of two sections of the same mathematics course—Mathematics for Elementary School Teachers—taught by the same instructor. PSTs comprise a special population of participants for the study because they practice metacognitive skills in the context of solving elementary school mathematics tasks. Specifically, they are encouraged to explain their reasoning when solving tasks. Twelve PSTs agreed to participate in the interviews.

Interviews lasted about 75 minutes and occurred in three parts: an assessment of their available structures for coordinating units (e.g., two-level structures like the one shown on the right side of Figure 1) using interview tasks from prior studies with middle school students (Norton et al, 2015); an assessment of working memory using backward digit span with digits read aloud (Morra, 1994); and the fractions tasks. Using the units coordination and working memory assessments from the first two parts of the interview, a subset of ranked fractions tasks was selected for each PST. Tasks were ranked by cognitive demand based on the number of unit constructions and transformations that might be required to solve the task, without reliance on units coordinating structures. We intended for initial tasks to impose low cognitive demand on PSTs and for later tasks to impose high cognitive demand, so generally, PSTs assessed with lower M-capacity began with simpler (lower ranked) tasks. Table 1 presents the four tasks on which we focus in this report.

Table 1: Fractions tasks (adapted from Hackenberg & Tillema, 2009)

Task #	Rank	Task Description
5	8	Imagine this [drawing a rectangle] is 5/9 of a whole candy bar. So, how could you make 1/9 of the whole candy bar from what you have?
6	10	Imagine a rectangular cake that is cut into 15 equal pieces. You decide to share your piece of cake fairly with one other person. So, how much of the whole cake would that person get?

8	12	Imagine you are at a party and a cake is cut into nine equal pieces. Two people show up to the party late and you decide to share your piece of cake with them. So, what fraction of the whole cake do the latecomers get together?
10	14	Imagine cutting off 1/4 of 5/6 of a cake. So, how much is that of the whole cake?

Data Analysis

Data analysis reported here consists of real-time and retrospective analysis of PSTs' responses to the fractions tasks. During the interview, we assessed PSTs' abilities to solve the tasks without using figurative material in order to determine whether to continue to more challenging (higher ranked) tasks. The interview continued with higher and higher ranked tasks until we inferred that the PST was unable to produce correct or confident solutions. In some cases, PSTs were explicit about their own perceived limitations; e.g., "I have no idea" or "my brain is confused now."

After all interviews were completed, the team began retrospective analysis of the behavioral video data. The videos were analyzed, PST by PST, moving from lowest ranked to highest ranked tasks. For each task, the video analysis consisted of two main parts: the first consisted of classifying the demand of the task for the students, and the second consisted of creating a UTG for the PST's actions in solving the task. A constant comparative analysis was used throughout both parts of analysis to promote consistency.

Analysis of the videos was done together by the research team with at least two of the three team members present. Each task was assigned a classification for the cognitive demand of the task, based on behavioral indicators during the PST's response to the task. Cognitive demand was coded as Low, High, or Over, depending on how challenging the task seemed to be for the PST in managing the units and unit transformations involved in solving the tasks. The Low code indicates that the PST's response was quick and confident. The High code was used when the PST struggled presumably operating near the limits of their M-capacity, as indicated by expressed doubt, rehearsing the task's solution, and requests for the task to be repeated. The Over code indicates that the task was beyond the students' ability to solve without figurative material or help from the interviewer.

Once cognitive demand codes were assigned for each task, the research team went back and watched the video again in order to build a UTG for each task's solution, illustrating the sequence of actions (operations) the PST used to reach a solution. The graphs serve as explanatory models for the PSTs' observed behavior in solving tasks by drawing on cognitive resources. Using the constant comparative method, we iteratively returned to prior graphs to ensure the models were consistent across tasks and PSTs. Adjustments were made to prior graphs as new features emerged in newer graphs.

Results

We chose to focus on PST 22 because she was one of two PSTs with the highest assessed M-capacity (7), and of those two, was the only PST operating at the lower stage of units coordination (constructing two-level unit structures but not three-level unit structures). Here, we analyze her responses to the four tasks shown in Table 1.

Task 5

PST 22 exhibited Low cognitive demand in solving Task 5. As soon as the task was posed, she responded, "well if you have five-ninths of it, taking away four ninths would give you one-ninth because five minus four is one." The UTG shown in Figure 2 illustrates the mental actions we inferred PST 22 used in solving the task.

PST 22 seemed to rely on a part-whole understanding of fractions. Rather than structuring 1/9 as a one-to-nine size relation between the 1/9 part and the whole, PST 22 seemed to partition the whole into nine parts (P_9) and, reversibly, take their unitized collection as the whole (U_9). The PST needed to disembed five of those parts from the whole (D_5) to establish 5/9 as five parts out of nine equal parts in the whole. Then, taking away four of those five parts, through a second use of disembedding (D_1), would leave 1/9 as one out of nine equal parts in the whole. As such, PST 22 would experience Task 5 as having an M-demand of 3, well below her M-capacity of 7.

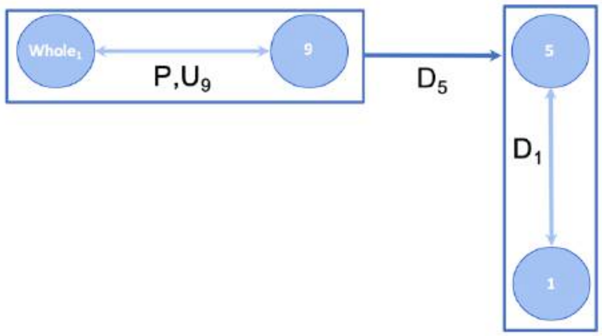

Figure 2: Making a unit fractional part from a non-unit fractional part (Task 5)

Tasks 6

Tasks 6 involved finding a fractional value (relative to the whole) when taking a unit fraction of a unit fraction. With the composition of two fractions, these tasks should impose additional cognitive demands, relative to Task 5. These increased demands are indicated in the PSTs' behavioral responses to the tasks, but behavioral indicators did not meet the threshold of High cognitive demand and, so, we categorized them as Low.

PST 22's response to Task 6 was immediate: "You get one fifteenth of the cake and split that in half. My first thought was one-thirtieth of the cake, because [makes splitting motion with hands in the air] splitting that in half, like if you were to split every piece of fifteen in half, then that would be like one thirtieth of the entire case." She seemed to imagine partitioning each one of the fifteen original parts into two parts to produce 30 parts in the whole. This mental action aligns with the distributing operation ($T_{2:15}$), but the production of 30 equal parts would be essential for PST 22's understanding of 1/30 as one out of 30 equal parts in the whole. We take such responses as indication PST 22 used her units coordinating structures to assimilate fractions as parts out of wholes. As indicated by the UTG (Figure 3) PST 22 experienced an M-demand of 4 for this task.

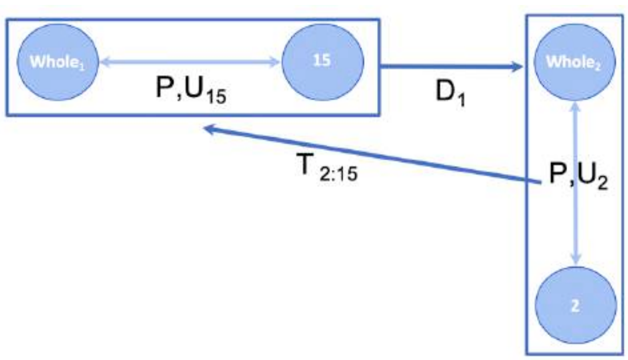

Figure 3: Finding a unit fraction of a unit fraction (Task 6)

Task 8

With the introduction of a non-unit fraction, Task 8 would also introduce one more unit and one more action to coordinate, increasing cognitive demand by 2 over Task 6. Indeed, the UTG for PST 22's response to Task 8 (not shown) would be structurally identical to the one for Task 6 (Figure 3), except it would include an additional action of disembedding two units from 27 (D_2) and the resulting unit of 2/27. Increased demand became evident in PST 22's response, which we took as indication of High demand.

PST 22: So, it's split up into nine equal pieces. So, then, you would split one ninth into…Two people come, but you still have a little bit? So, that… So, you would split that up into three. So, then I… Well, I guess you would do one ninth times two thirds to get how much they equal, like how much both their pieces would be. And then whatever that is, I guess it would be… two over… two eighteenths? Wait, that doesn't seem right. [pauses for five seconds] I feel like… I mean, I guess… You take those nine pieces, splitting that one ninth into thirds. But to find out how much two of those thirds are, you'd multiply one ninth by two thirds… Or no. You'd… you'd multiply the one ninth by one third, and then just do that twice? I don't know if that'd give you the same answer.

Researcher: Okay. Uh, let's… Maybe I can help you.

PST 22: Okay.

Researcher: If you want me to be your calculator again, I'll do it.

PST 22: [begins to draw on table with finger] So, you do one ninth, which divided by three, so you could times it by one third. So, then you'd have one over um… [pauses for five seconds.] Oh wait… [whispers to self] Three times nine, that's twenty-seven. Oh no, one over twenty-seven. And then you multiply that by two… to get two-thirds or to get two parts of the thing… So, then I guess… What's one over twenty-seven times two? Is that just two-twenty-sevenths? Okay.

Researcher: Nice, I like the way you reasoned through it. Yeah.

PST 22: Okay. I was like, because I was thinking one over twenty-seven times two over one and I was like I guess that's just two, twenty-sevenths.

This response indicates that PST 22 tried to rely on the standard algorithm for multiplying fractions but struggled to reconcile it with prior reasoning. She began as she had in Task 6, partitioning the whole, disembedding one of those parts, and then partitioning it into smaller parts. However, in contrast to Task 6, she then began referring to a fraction multiplication, 1/9 times 2/3. Fraction multiplication might have helped her keep track of the additional unit involved in this task (the 2 in 2/3), but she was not sure that the multiplication of fractions would generate the correct result. Her concerns were heightened when she mistakenly multiplied 9 times 2, instead of 9 times 3, to produce two eighteenths: "that doesn't seem right." So, she reverted to operating on the 9 units, partitioning them into thirds, which she was then able to reconcile with 1/9 times 1/3. Thinking of the task as the multiplication problem, 1/9 times 2/3, then did work for her by maintaining the 2 in two-thirds: "and then you multiply that by 2 to get two-thirds, or to get two parts of the thing."

We found PST 22's persistence in response to Task 8 impressive and indicative of her high M-capacity. Ultimately, she reasoned with parts out of the whole ("two parts of that thing"), as she had before, but was able to meet the cognitive demands of the task by organizing her operations around the fraction multiplication algorithm. So, while the task was highly demanding for her, we see evidence for how algorithms, when made meaningfully related to (or reconciled with) operations, can offload the demands of mathematical reasoning.

Task 10

For Task 8, the PSTs needed to distribute the new partitioning (thirds) across the nine units making up the whole, and then take two of the resulting parts (1/27ths). For Task 10, she needed to distribute fourths across both the five parts in five-sixths and the six parts in the whole. The increased demands caused PST 22 to lose track of the six parts making up the whole. Figure 5 presents the UTG for this response.

PST 22: You're cutting off one fourth of five sixths of a cake?
Researcher: Yes.
PST 22: [uses hands to show number of pieces on the desk and begins talking to self] So, you'd have, so you'd have six pieces…and out of those five…you want to cut off one fourth of that. Um…I guess you would…I mean I guess you could split those five pieces into four and get one of those, but I'm trying to think like numbers-wise what that would...I. well… [pauses for seven seconds] I guess of those five pieces you could…Split them into…Like you could get a…Split them into twenty pieces because five times four is twenty and then, um, you would take one fourth of that… I guess it would be five pieces. Yeah, it would be five pieces of that twenty to find the one fourth of the five sixth. Is that, do I need to explain it more?
Researcher: Okay, uh let's…
PST 22: Which would be, do you want me to draw it? [reaches towards paper]
Researcher: Well tell me the final answer and then we can draw it.
PST 22: Um, oh gosh it would be… [pauses for four seconds] Splitting twenty, it would be five…Well it would be five twentieths, which would equal one fourth, so like five of those, but then I don't know how to figure that out into sixths. I think that's my…
Researcher: Yeah that's cool, I like the way you're reasoning. Let's draw it, and I think you will figure it out.

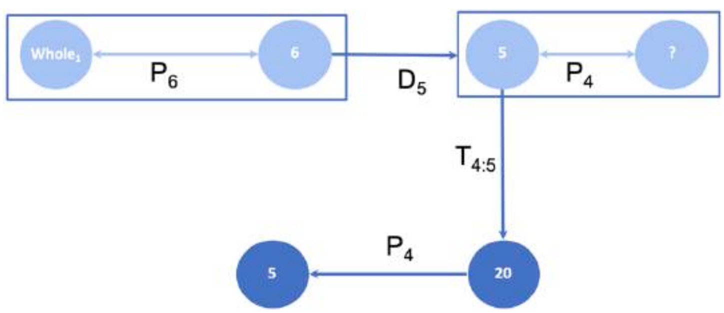

Figure 4: Finding a unit fraction of a non-unit fraction (Task 10)

Once again, PST 22 seemed to conceptualize the initial fraction (5/6) as a part-whole relation, partitioning the whole into six equal parts and disembedding five of them. This inference is supported by the PST's verbalization, "so you'd have six pieces…and out of those five." As PST 22 tried to find one-fourth of 5/6, she operated only on the five disembedded parts and lost track of the sixth part making up the whole. Thus, rather than fourthing 5/6, as originally intended, she ended up distributing four parts into each of the five parts, producing 20 parts, and taking one-fourth of those 20 parts instead.

Discussion

In line with the stated goals of PME-NA, we aimed to elucidate the psychological aspects of learning mathematics. Specifically, UTGs integrate the psychological construct of working memory with a construct from mathematics education research—units coordination—to explain how mathematics arises through the coordination of students' own mental actions (Beth & Piaget, 1966). The case study of PST 22's solutions to a ranked set of fractions tasks demonstrate the explanatory power of UTGs.

With an assessed working memory of 7, PST 22 would have experienced even Task 5 as highly demanding; her solution to the task involved four units and three transformations between them. However, PST 22 assimilated some of those units and transformations into two-level structures, thus chunking them into single two-level units so that she experienced Low cognitive demand. As a theoretical construct, chunking has its roots in cognitive psychology (e.g., Pascual-Leone, 1970), but UTGs illustrate how chunking can take a particular form in mathematics, as units coordinating structures (Steffe, 1992; Hackenberg, 2007).

Offloading cognitive demand with unit structures afforded PST 22 the ability to account for additional units and transformations in progressively more demanding tasks. In looking at the UTGs across the four tasks (as illustrated in Figures 2-4), we see how PST 22 introduced additional transformations (such as distributing) and units to solve Tasks 6 and 8, until she reached her threshold, with 7 units/transformations. Having reached that threshold, she was not able to proceed in solving Task 10, but we might imagine ways that she might further structure units into chunks to increase her mathematical power.

Acknowledgments

We thank the Adaptive Brain & Behavior Destination Area at Virginia Tech for their support of this project. We also thank undergraduate researcher, Rachael Stryker, for her contributions.

References

Agostino, A., Johnson, J., & Pascual-Leone, J. (2010). Executive functions underlying multiplicative reasoning: Problem type matters. *Journal of Experimental Child Psychology*, *105*(4), 286-305.

Beth & Piaget (1966). *Mathematical epistemology and psychology* (Trans. W. Mays). New York: Gordon and Breach.

Blankenship, T. L., Keith, K., Calkins, S. D., & Bell, M. A. (2018). Behavioral performance and neural areas associated with memory processes contribute to math and reading achievement in 6-year-old children. *Cognitive development*, *45*, 141-151.

Boyce, S., & Norton, A. (2016). Co-construction of fractions schemes and units coordinating structures. *The Journal of Mathematical Behavior*, *41*, 10-25.

Boyce, S., & Norton, A. (2019). Maddie's units coordinating across contexts. *The Journal of Mathematical Behavior*.

Bull, R., & Lee, K. (2014). Executive functioning and mathematics achievement. *Child Development Perspectives*, *8*(1), 36-41.

De Smedt, B., Janssen, R., Bouwens, K., Verschaffel, L., Boets, B., & Ghesquière, P. (2009). Working memory and individual differences in mathematics achievement: A longitudinal different levels of representing unknowns. *Journal for Research in Mathematics Education*, 52-80.

Hackenberg, A. J. (2007). Units coordination and the construction of improper fractions: A revision of the splitting hypothesis. *The Journal of Mathematical Behavior*, *26*(1), 27–47.

Hackenberg, A. J. (2013). The fractional knowledge and algebraic reasoning of students with the first multiplicative concept. *The Journal of Mathematical Behavior*, *32*(3), 538-563.

Hackenberg, A. J., & Tillema, E. S. (2009). Students' whole number multiplicative concepts: A critical constructive resource for fraction composition schemes. *The Journal of Mathematical Behavior, 28*(1), 1-18.

Morra, S. (1994). Issues in working memory measurement: Testing for M capacity. *International Journal of Behavioral Development*, *17*(1), 143-159.

Norton, A., Boyce, S., Phillips, N., Anwyll, T., Ulrich, C., & Wilkins, J. L. (2015). A written instrument for assessing students' units coordination structures. *Mathematics Education*.

Pascual-Leone, J. (1970). A mathematical model for the transition rule in Piaget's developmental stages. *Acta Psychologica*, *32*, 301-345.

Simon, M. A., Placa, N., Avitzur, A., & Kara, M. (2018). Promoting a concept of fraction-as-measure: A study of the Learning Through Activity research program. *The Journal of Mathematical Behavior*, *52*, 122-133.

Steffe, L. P. (1991). Operations that generate quantity. *Learning and individual differences*, *3*(1), 61-82.

Steffe, L. P. (1992). Schemes of action and operation involving composite units. *Learning and individual differences*, *4*(3), 259-309.
Steffe, L. P., & Cobb, P. (2012). *Construction of arithmetical meanings and strategies*. Springer Science & Business Media.
Steffe, L. P., & Olive, J. (2010). *Children's fractional knowledge*. New York: Springer
Swanson, H. L., & Beebe-Frankenberger, M. (2004). The relationship between working memory and mathematical problem solving in children at risk and not at risk for serious math difficulties. *Journal of Educational Psychology*, *96*(3), 471.
Tillema, E. S. (2013). A power meaning of multiplication: Three eighth graders' solutions of Cartesian product problems. *The Journal of Mathematical Behavior*, *32*(3), 331-352.
Tzur, R., & Simon, M. (2004). Distinguishing two stages of mathematics conceptual learning. *International Journal of Science and Mathematics Education*, *2*(2), 287-304.
Ulrich, C. (2016). Stages in constructing and coordinating units additively and multiplicatively (Part 2). *For the Learning of Mathematics, 36*(1), 34-39.
Wilkins, J. L. M., & Norton, A. (2011). The splitting loope. *Journal for Research in Mathematics Education, 42*(4), 386-406.Wilkins, J. L. M., Norton, A., & Boyce, S. (2013). Validating a written instrument for assessing students' fractions schemes and operations. *The* Mathematics *Educator, 22*(2), 31-44.