UNDERSTANDING THE ROLES OF PROOF THROUGH EXPLORATION OF UNSOLVED CONJECTURES

Jeffrey Pair
California State University Long Beach
Jeffrey.pair@csulb.edu

Gabe Calva
California State University Long Beach
Gabriel.calva@student.csulb.edu

Mathematics majors, including future secondary teachers, should understand the work of pure mathematicians and the crucial role proof plays for the discipline of mathematics. Beyond the textbook proofs seen in most transition-to-proof courses, we conjectured that students might develop a deeper understanding of the discipline of mathematics and proof if they had the opportunity to do mathematical research—to try and prove an unsolved conjecture. As an added component of our transition-to-proof course, we designed an intervention so that students researched the Twin Primes conjecture or the Collatz conjecture. Students wrote reflections about their research and described how their perceptions of mathematics were influenced by the research. We analyzed the reflections and sought to understand how the students' views of mathematics and proof were enriched, if at all, through research on unsolved conjectures.

Keywords: Reasoning and Proof; University Mathematics; Problem Solving; Affect, Emotions, Beliefs, and Attitudes

Mathematics teachers should have a rich understanding of the discipline of mathematics. When a teacher possesses a healthy and informed conception of the discipline, she is well-positioned to pass on productive beliefs about mathematics to her students. Teachers and students benefit from understanding the important thought processes and practices that make mathematics the unique and enjoyable field that it is. But many mathematics majors and school teachers have naïve views of the discipline of mathematics (Pair, 2017; Thompson, 1992). Pair (2017) found that students in a transition-to-proof course were not familiar with conjectures in general and generally unaware of well-known conjectures such as the Twin Primes conjecture.

Students' Understanding of the Function of Proof

Research has shown that many students, even mathematics majors, struggle learning to prove (Bleiler, Thompson, & Krajčevski, 2014). This may be due in part to the fact that students do not experience the functionality of proof in the same way that mathematicians do (de Villiers, 1990). For instance, proof for a mathematician can a be a means to convince other mathematicians that a claim is true. Students may not experience proof in this way if they are only asked to prove textbook exercises which are assumed true from the get go.

De Villiers (1990) described five roles of proof for mathematicians that he conjectured would be productive for students to understand and experience: 1) Verification: Proof serves as a means of knowledge justification that enables mathematicians to obtain conviction that a claim is true; 2) Explanation: Proof provides insight into why a mathematical claim is true; 3) Systematization: Proofs serve to organize the deductive system of axioms, definitions, and theorems; 4) Discovery: Mathematicians make new discoveries through proof; and 5) Communication: A proof is a means by which mathematicians communicate mathematical knowledge.

We believe that a deeper understanding of the roles of proof corresponds to a deeper understanding of the discipline of mathematics. Some researchers have documented the types of course activities (e.g. critiquing classmates' proofs) that may engage students in these roles of proof (Bleiler-Baxter & Pair, 2017; Cilli-Turner, 2017). We conjectured that exploring unsolved conjectures may also

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provide students an opportunity to deepen their understanding of the discipline of mathematics and experience the functionality of proof as mathematicians do.

Transition-to-Proof Intervention Research Study

We conducted a research study in a transition-to-proof course at a university in Southern California. The course was required for students majoring in either pure mathematics or secondary mathematics education. There were thirty-three students enrolled in this class. Twenty-five of the students agreed to participate in the research project which was approved by our University's Institutional Review Board. Our University is a Hispanic-Serving Institution, and a diversity of races and genders are present in our sample.

As an added component of the course, students explored and invented their own methods to navigate one of two famous mathematical conjectures: either the Twin Primes conjecture or the Collatz Conjecture. Typically attributed to Euclid, the Twin Primes conjecture states that there are an infinite number of twin primes (Rezgui, 2017). Two prime numbers $x < y$ are twin primes provided $y = x + 2$. For instance, 3 and 5 is the smallest twin prime pair. The Collatz conjecture, named for the German mathematician Lothar Collatz, is also known as the $3n + 1$ problem (Bairrington & Okano, 2019). The conjecture concerns an iterative process on natural numbers. Given any natural number n, if n is odd multiply by 3 and add 1; or if n is even, then divide by 2; repeat the process on the resulting natural number; repeat again. The conjecture is that for any natural number, this iterative process will eventually reach 1. For instance, if we take n=3 the corresponding Collatz sequence is 3, 16, 8, 4, 2, 1 (if continued the sequence would cycle 4,2,1, 4,2,1…). These two conjectures have remained unsolved to the present day.

Students in the transition-to-proof course explored these conjectures for an entire semester, documenting their work and reflections in what was called their *mathematicians' notebooks*. Their work in the mathematicians' notebooks accounted for 5% of their course grades. In the first notebook assignment, students were tasked with exploring both the Twin Primes conjecture and the Collatz conjecture. For subsequent assignments, students chose which conjecture they would like to explore. Students were assigned to research teams with other members of the class based on their conjecture preferences.

About half of the students worked on the Collatz Conjecture[1] and half on the Twin Primes conjecture, with some students exploring both. Midway through the semester the students shared their findings with other members of the class. Students drew inspiration from each other and adopted their classmates' approaches in subsequent assignments. Twice during the semester, the instructor collected students' notebooks and provided feedback and direction to guide students in their explorations. One assignment also required the students to watch and reflect on videos of mathematicians addressing their own work on the conjectures.

See Figure 1 for an example of student exploration on the Collatz conjecture from a student's notebook. This student, a Hispanic male, was working backwards from 1, trying to show that all numbers will eventually cycle to 1. His first line shows powers of 2, which obviously will reach 1. For each power of 2, he considered if it was of the form 3n+1 by subtracting 1 from the number and dividing by 3. When he found such a number, he would find the value of n and use it to start a new number line—which started with n and increased by a factor of 2. He wrote "[the pink] represents how a new line in the form of $n \cdot 2^{x}$, where $x \geq 0$ and n is an odd integer, is made by subtracting by 1 and then dividing by 3 by another line in the same form. [The green line] represents the set of natural numbers in order starting from 1." This student recognized that if he could successfully show

[1] Slightly more students worked on the Collatz conjecture than the Twin Primes conjecture.

that every natural number appears in one of the number lines, then he could prove the Collatz conjecture.

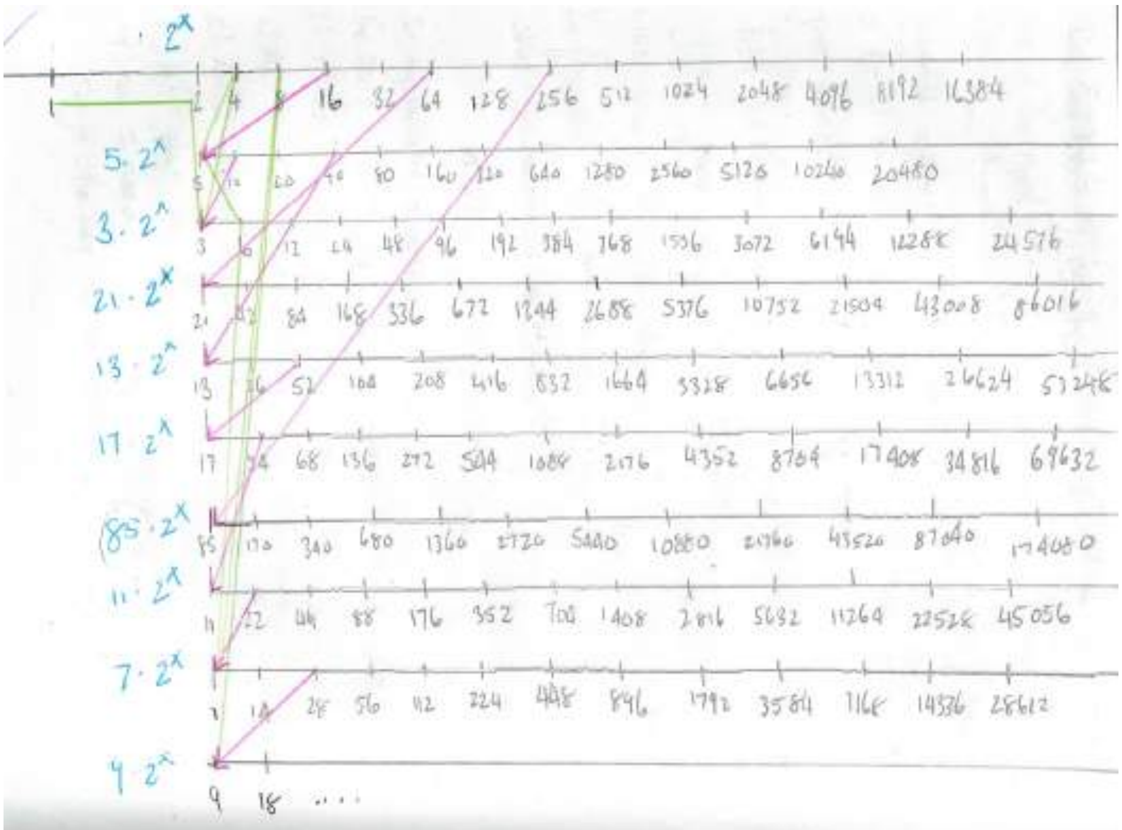

Figure 1: Example of Student Exploration of the Collatz Conjecture

We conjectured that through work on unsolved conjectures, mathematics majors would have novel opportunities to enrich their understanding of the nature of mathematics and the roles of proof. The first author (instructor of the course) designed the notebook assignments so that they would be a source of data useful for providing insight into what the students were learning about the nature of mathematics and proof as they explored the conjectures. Students had several reflective prompts they were required to address in their notebooks. For instance, at the beginning of the semester, students responded to the prompts *What is mathematics all about?* and *What is a mathematical proof, and how is it used by mathematicians?* The students responded to these prompts again at the end of the semester; they also responded to two additional prompts: (1) *How has your thinking regarding mathematics and mathematical proof developed and changed during this semester? Which changes were the result of engagement in the standard course activities (e.g. homework, tests, lectures) and which changes were the result of your experience conducting mathematical research (working on the unsolved conjectures)?* and (2) *What were the challenges and successes of your experience with mathematical research this semester?* The researchers analyzed student reflections to understand students' experiences working on the conjectures. In particular, we analyzed the notebook data to discern what students learned about the nature of mathematics and the roles of proof through their research.

Data Analysis

The data for the research study consisted of 25 student notebooks. The instructor (first author) and an undergraduate student from the course (second author) analyzed the data. First, each researcher individually read and examined a particular students' notebook. As we individually read through each student's notebook we took note of 1) Instances where students reflected about the nature of mathematics; 2) Possible evidence for changes in student's perceptions of the nature of mathematics (especially in regards to the five roles of proof); 3) Interesting mathematical ideas and approaches the student generated in working on the conjectures; and 4) Other interesting ideas expressed by the students. Each researcher then wrote a summary of his individual findings for the particular student's notebook.

After reading and analyzing a notebook, the researchers then shared their thoughts and findings with each other. We made notes on insights gained from the others' analysis and made note when a student demonstrated an understanding of any of the five roles of proof. We then repeated the process for the next student notebook.

Once we had analyzed all of the individual notebooks, the next step was for each researcher to create a holistic summary of the data that included broad themes in the students' responses as well as evidence (from the data) to back up our claims. The creation of these holistic summaries involved sorting student quotations into categories or themes (Ryan & Bernard, 2003), providing evidence that students' understandings of the nature of mathematics seemed to be enriched through the notebook project, and tallying how many students expressed certain ideas regarding the nature of mathematics in regards to the Roles of Proof framework. We also identified other recurring themes in the data related to students' experiences. We then shared our findings with each other, challenging each other to provide evidence for claims, which led to further refinement of the findings. We now present the results.

Results

We found that students alluded to four of de Villiers' (1990) five roles of proof while participating in this study: verification, explanation, systematization, and discovery.

Verification

Early in the semester, eight students' descriptions of the purpose of proof alluded to the notion of verification. For instance, a student wrote "A mathematical proof is a tool mathematicians use in order to determine if their statement is true or false." But by the end of the semester, seven additional students described the role of verification. These students wrote either about the role of proof in convincing others that a theorem is true, or the importance of proof in justifying and validating mathematical claims. For instance, one student wrote "Ideally, [a proof] should have no errors (holes) and must convince other mathematicians that it is correct and true for its purpose." Students were well aware of their inability to find a convincing proof for either the Twin Primes conjecture or the Collatz conjecture.

Explanation

Only two students alluded to the role of explanation in their initial descriptions about the purpose of proof, but by the end of the semester nine additional students alluded to this role of proof in their reflections. These students used language that emphasized proof's role in providing insight into why a mathematical claim is true. For instance, at the end of the semester a student wrote "At first, I feel like mathematics was more of just applying formulas and theorems to solve problems, but over the course I learned that it is more important to know why theorems work and how they work." Other students described how proof was needed to understand mathematics deeply. At the end of the semester a student wrote, "A mathematical proof is a way to understand how and why certain mathematical concepts exist. In math, there is always a reason for everything and math proofs help explain those reasons further. It is used by mathematicians to understand problems further and on a deeper level." Students recognized the surface level simplicity of the two conjectures (some students described how even an elementary student could understand them) but also recognized the complexity of what was required to understand *why* the conjectures were true.

Systematization

Four students alluded to the systematization role of proof at the beginning of the semester, and three additional students alluded to systematization at the end. These students described a building up of mathematics—results serve as the foundation for future results. A student wrote, "Proofs are facts so they have a crucial part to play in the development of other proofs where progression in the proof has more difficulty than normal." And another student, referencing a video she watched related to the twin primes, wrote, "Mathematicians use these proofs to help prove other conjectures. As Maynard's proof was influenced by Zhang's proof. Eventually, Maynard's proof will be used to help prove other conjectures." These students came to see that mathematicians work on conjectures; and proofs are used to build the body of mathematical knowledge.

Discovery

Four students alluded to the discovery role of proof, all at the end of the semester. These students seemed to better understand the mathematician's quest to discover new results. One student wrote, "Proofs are used by mathematicians to assist them in creating other proofs to eventually have a breakthrough that is groundbreaking in mathematics as well as the world." Another student wrote, "Mathematics is all about solving the world's greatest mysteries. Just from taking this class I have learned that mathematicians discover new problems and then spend their life trying to prove/understand it." See Figure 2 for a display of how many students alluded to a role of proof in their reflections.

Role of Proof	Alluded to at the Beginning of the Semester	Alluded to by the End of the Semester (but not at the Beginning)
Verification	8	7
Explanation	2	9
Systematization	4	3
Discovery	0	4
Communication	0	1

Figure 2: Number of Student Reflections on the Role of Proof

Other Results

We believe that most of the students had naïve views about pure mathematics and what it entails for mathematicians as they began the course. The notebook assignments provided students the opportunity to try their hand at proving conjectures that even famous mathematicians have not yet proven. This gave many of the students the opportunity to develop new ideas about what the discipline of mathematics and proof is all about, as many of them were dealing with a type of mathematics problem that they had never even conceived.

The notion of an unsolved conjecture provoked some of the students to start thinking more deeply about the topic, which in-turn led to creative and outside-the-box ideas. Eight of the students either showed a creative process in solving the conjectures, or stated they learned they could be more creative/look at conjectures from different angles and perspectives. Some forms of this creativity included working backwards on the Collatz conjecture, creating code to help find patterns and links within the Collatz conjecture, as well as finding a formula for the distance between twin prime pairs.

One student stated in their last reflection that "I learned we can look at an idea from many different angles." This student learned that some mathematical problems, such as conjectures, are not always a one-way road.

Another common trend amongst the students was that in the reflections, a number of the students took a step back and reflected from a broader point-of-view, and said they developed a deeper understanding of the nature of mathematics. Some students stated that they get the "bigger picture," or that they understand what happens "behind the scenes of mathematics." Of the 25 students, 10 of them had reflections in this vein. A few of those students were also ones that had a narrow idea of what mathematics was about in the beginning of the semester. These students originally had a concrete perception of mathematics being about calculations. For instance, a student wrote, "to me, mathematics is mainly about obtaining problem-solving skills through various mathematical problems," and, "mathematics is all about using formulas to solve problems." Working on an unsolved conjecture may have had an effect on their idea of mathematics and showed them that there is more to mathematics and it is not always about performing algebra with numbers. The same student, in their last reflection, wrote that "mathematics is all about solving the world's greatest mysteries."

The students' approaches and their perceptions of mathematics were not the only things they reflected about. Some of the students described a variety of emotions in their reflections. Eight of the 25 students expressed enjoyment in regards to the notebook. One student expressed how "math is getting more and more creative." Another student even stated that they were reassured and glad to be a mathematics major after completing the notebook, writing "I made the right choice in majoring in math since I love proofs so much."

Challenges

One of the most commonly noted student challenges was the struggle with finding where to begin proving the conjectures. Some of the students seemed overwhelmed at the prospect of exploring an unsolved conjecture that many mathematicians have tried and failed to solve. Some even expressed misconceptions, believing that they were being asked to prove something "impossible." Others were confused how they were supposed to go about proving something "unprovable." One student wrote, "I think the main challenge is just the fact that it's a conjecture. I could not find a way to write any proof because I did not understand or recognize the pattern behind it. I'm not sure if I'm doing the assignment correctly." Other students had a defeatist attitude, not believing that they would have anything positive to contribute: "Working on the conjecture was more irritating than exciting because I can't prove it. It took away all the satisfaction because it is a famously unproven conjecture and I couldn't solve it." Another student wrote, "Although it was intriguing, I gave up preemptively because I knew that I would not do anything that would help come to any conclusion." Although these students had trouble making progress, most were able to engage with the conjectures in some way. In subsequent semesters when implementing this project, the first author has incorporated more in-class discussion time for the conjectures. This has allowed the students more opportunities to get ideas from their peers for how to approach the conjectures.

Conclusions

Overall, we are encouraged that exploration of the unsolved conjectures was a productive experience for most students, and helped them to understand the role of proof and the discipline of mathematics in novel ways. We found that students had the opportunity to learn about what Hersh (1991) referred to as "the back of mathematics"—the messy informal work involved in proving conjectures. We believe that some students experienced and understood proof's role as verification, explanation, discovery, and systematization. Of the five roles of proof, explanation was most reflected on, with 9 students alluding to this role in their reflections (beyond the 4 that alluded to it at the beginning of the semester). We believe that working on unsolved conjectures provided students a special opportunity to understand this role of proof, as they were forced to come to the terms with the fact that even when they believe a conjecture to be true, it is not always easy to understand why it is true. We note that of the five roles of proof, communication was the only one we could not find addressed in the student reflections.[2] Other researchers have found that small-group work in inquiry-based classrooms brings to the forefront the communication role for students (Bleiler-Baxter & Pair, 2017; Cilli-Turner, 2017). Perhaps communication was not discussed by the students in our study as only one classroom day was devoted to student discussions of their work on the conjectures. We believe allotting more time for classroom discussion of conjectures may help students better understand the communication role of proof.

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[2] One student did address the communication role of proof; but she had previously participated in a summer research project exploring an unsolved conjecture, and thus had additional opportunities to learn about the importance of communication in mathematics.