DEVELOPING A FRAMEWORK FOR CHARACTERIZING STUDENT ANALOGICAL ACTIVITY IN MATHEMATICS

Michael D. Hicks
Texas State University
mh1653@txstate.edu

This report proposes a framework for describing student analogical reasoning activities in abstract algebra that moves beyond the traditional literature-based treatment of analogical mapping. The Analogical Reasoning in Mathematics (ARM) framework captures the activities that students engage in when anticipating, creating, and reasoning from mathematical analogies. This considers activities along several dimensions including: inter/intra domain activity, foregrounded/backgrounded domain, and attention to similarity/difference. These dimensions are integrated with Gentner's (1983) analogical mapping framework to characterize student activity when they are presented with tasks where reasoning by analogy can productively support their mathematical investigations. By characterizing these activities, we can better develop tasks to support students in productively analogizing between mathematical domains.

Keywords: Advanced Mathematical Thinking, Algebra, Analogical Reasoning

Throughout history, analogies have played a vital role in the development of key mathematical concepts and connections such as Descartes' recognition of the analogical similarities between algebra and geometry (Crippa, 2017). In modern mathematics instruction, analogies have been argued to be useful in developing student conceptual understanding by assisting students in utilizing prior knowledge to make sense of new contexts and develop conceptual understanding rich in connections across mathematical domains. However, unguided analogical reasoning may result in unproductive mathematical reasoning (Sidney & Alibali, 2015). By investigating the nature of students' mathematical reasoning as they develop and reason from analogies in mathematical contexts, we can begin developing support for students to productively reason by analogy.

Outside of mathematics education, Gentner (1983) introduced the Structure-Map Theory (SMT) to describe analogical reasoning as mappings across domains. In particular, this notion of mapping requires attention to similarity across domains. This type of focus is consistent with much of the theories that followed (e.g. Holyoak & Thagard, 1989). Attending to similarity is crucial for generating analogies. Surface similarities provide an access point for the generation of analogies (Holyoak & Koh, 1989). These components of analogical reasoning are shown in Figure 1 below.

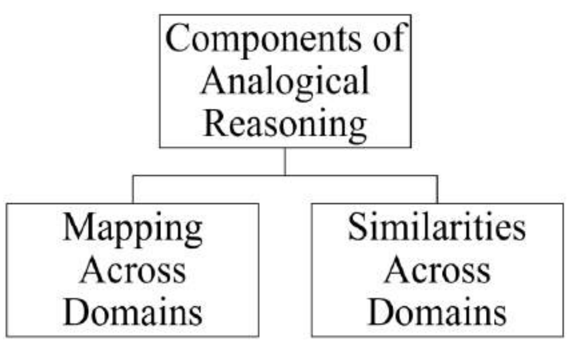

Figure 1: Components of Analogical Reasoning

Although the framework above is useful for categorizing a particular analogical mapping, there exist nuances of analogical reasoning within mathematical thinking that are not explicitly captured by

In: Sacristán, A.I., Cortés-Zavala, J.C. & Ruiz-Arias, P.M. (Eds.). (2020). *Mathematics Education Across Cultures: Proceedings of the 42nd Meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education*, Mexico. Cinvestav / AMIUTEM / PME-NA. https:/doi.org/10.51272/pmena.42.2020

this framework. In this paper, I present the Analogical Reasoning in Mathematics (ARM) framework for characterizing the mathematical activity of students as they anticipate, create, and reason from analogies between mathematical domains. In particular, this framework expands upon the ubiquitous components of mapping across domains and attending to similarities, and introduces a new component of foregrounding a domain. This interpretive qualitative study seeks to contribute to answering the following research question:

What are the mathematical activities of students as they reason by analogy about structures between group theory and ring theory?

Theoretical Framing

I adopt Gentner's SMT as a foundation for developing a conceptual framework for identifying and describing analogical reasoning in mathematics. In particular, I borrow the concept of *domains* and the process of *mapping across domains* as a basis for identifying and describing analogical reasoning in mathematics. I define analogical reasoning as the act of identifying or conjecturing about a perceived correspondence between two (or more) domains. In addition, I define *analogical activity* as mathematical activity occurring within and around analogical reasoning.

A domain is a collection of knowledge held about a mathematical concept or situation. For example, one could reason specifically about the domain of two-digit addition problems, or more generally about the domain of binary operations. A key aspect of reasoning by analogy is to utilize knowledge in one domain in order to develop knowledge within another. This process occurs through mapping across domains. The domain from which knowledge originates is known as the *source domain*, while the domain to which source knowledge is being applied is the *target domain*.

I also adapt the content of mappings described by Gentner to this conceptual framework. Gentner proposed that three categories of content may be mapped between domains: objects, attributes and relations. Objects can be a single entity (e.g., a triangle), component parts of a larger object (e.g., the angles of a triangle), or coherent combinations of smaller objects (e.g., all equilateral triangles) An attribute is defined as a property or description of an object. This could include a definitional property of an object or a non-definitional descriptor of an object. Relations are properties that relate two or more objects, attributes or other relations together. A visual of these aspects in the context of an analogy between a ruler and a number line can be seen in Figure 2.

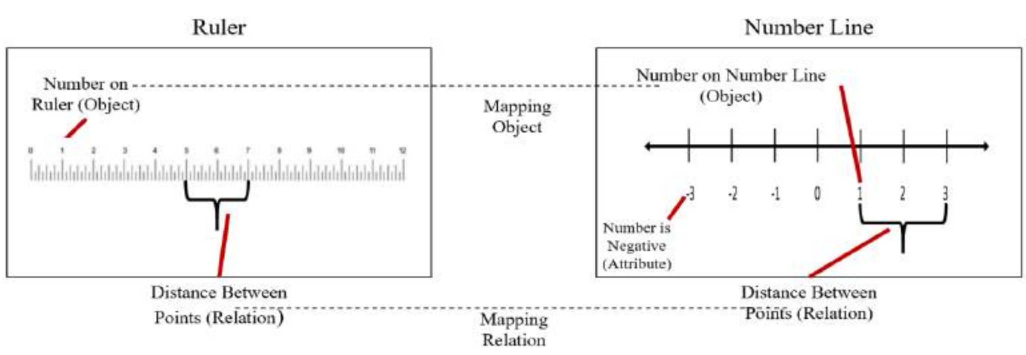

Figure 2: Mapping Between a Ruler and a Number Line

Methodology

Data Collection

The context of abstract algebra was chosen because of the existence of several naturally occurring structural similarities between group theory and ring theory. I conducted an initial pilot study with two mathematics students, one an undergraduate student in pure mathematics, and the other a graduate student pursuing a PhD in mathematics education. The participants in the present study included two undergraduate mathematics students who had previously taken a course in modern algebra emphasizing the theory of groups. Five 60-90-minute-long task-based interviews were conducted with the each participant. The initial interview helped to assess the participant's content knowledge of group theory before beginning to explore topics in ring theory. The ring interview provided the participants with the definition of ring and various tasks designed to acclimate the student to working with rings. The three subsequent interviews provided students with the analogizing task focused on reconstructing one of subrings, ring homomorphisms, or quotient rings by analogy with a structure in group theory. The interview tasks were constructed around three basic types: (1) Explicit elicitation of analogy generation, (2) example generation and checking (i.e., "give an example of a subring."), and (3) proof-writing (i.e., "Is the homomorphic image of a commutative ring commutative?") An example of a task meant to elicit explicit analogies is the following: *Make a conjecture for a structure in ring theory that is analogous to subgroups in group theory.*

Data Analysis

I used techniques outlined by Corbin and Strauss (2015) to analyze the data. First, the transcripts were segmented by identifying shifts in a particular mathematical idea or focus. Segments were identified by two criteria: (1) presence of analogical activity, and (2) shifts in mathematical focus. Each of the six interviews in which the analogizing task was given was coded for mapping activity and attending to similarity, as well as open coded for other aspects of analogical activity. Microanalysis was intermittently performed on segments when the nature of the analogical activity was unclear within a segment. Diagramming was incorporated to aid in making sense of how concepts fit together with one another. As a part of ongoing analysis, I wrote research memos to aid in explicating my thinking about concepts and generating new hypotheses. Results of microanalysis, diagramming, and memoing were regularly shared with colleagues to assist in ensuring the viability of my interpretations. Finally, the interviews focused on group theory and rings were used to help triangulate interpretations of the students' activity when possible.

From this process of coding and subsequent axial coding, two dimensions of activity related to analogical reasoning were identified in addition to the activity of mapping and attending to similarity, and one component of analogical reasoning was identified. These are each discussed at length in the following section.

Results

In this section, I share an overview of the Analogical Reasoning in Mathematics (ARM) framework with extended attention on categorizations that did not exist in the literature on analogy: intra-domain activity, and attending to differences. I then describe the component of foregrounding a domain which allows for deeper description of student mathematical activity while reasoning by analogy. Finally, I exhibit examples of mathematical activity during analogical reasoning that are characterized with the aid of the expanded framework.

The Analogical Reasoning in Mathematics (ARM) Framework

As shown in Figure 1, the literature on analogy is heavily focused upon mapping activity and attention to similarity. Within Gentner's (1983) framework, the heart of analogical reasoning is the process of mapping between domains. I refer to activity in which mapping occurs as inter-domain activity due to the nature of the activity as necessarily involving two or more domains along with activity occurring across those domains. However, there are times when a student may not engage with mapping activity. Instead, the student engages with activity that lies completely within a single domain. This category of activity is referred to as *intra-domain* activity. Intra-domain activities are those activities that operate within a single domain, either by focusing specifically on one domain, or by blending domains together. In addition, students may also attend to differences during the process of reasoning by analogy. Figure 3 exhibits the expanded framework for analogical reasoning including three components of analogical activity along with the dimensions of each component. Those marked with a '*' represent new aspects identified within the present study.

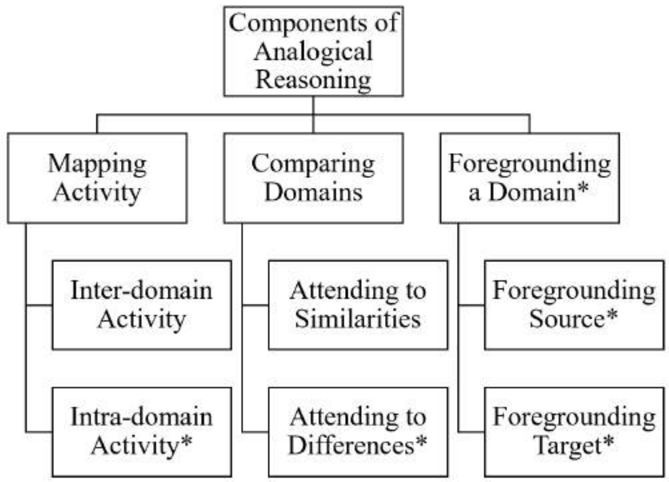

Figure 3: The ARM Framework

Foregrounding a Domain. The participants in this study were seen to place emphasis on different domains throughout the process of reasoning by analogy. This component of activity is referred to as *foregrounding a domain*. Consider the following quote from a student providing a rationale for their definition of what they call "normal subrings":

> Well I used the normal subgroup definition to apply it to normal subrings just because that act of using an operation I guess to apply it to your subring and making sure it's still and the ring itself is kind of the point of a normal subgroup so it's got to be important to use that for subrings or for normal subrings.

Within this first example, the student is justifying her definition of 'normal subring' by specifically pointing to the definition of normal subgroup and claiming that she "applied it to normal subring." The student is emphasizing the source domain as the motivation for her definition as is thus foregrounding the source domain of groups.

In contrast, consider the following exchange in which another student is reasoning about a proposed idea of normality in rings:

Student B: I don't know. We didn't talk about normal rings, so I didn't even think about that. I don't know. Well, with rings, you do have a lot more conditions than you do with groups, so I don't think I would need this normal on my ring because that's already included in my ring. So, I don't think I would have to mess with it.

Interviewer: Okay. Could you explain what you mean by that? Like you say, "It's already included in the ring."

Student B: Groups have four conditions or something to be called a group, and then rings have seven of them. So, since rings have more conditions, I think being normal is already one of those conditions, in a sense.

Within the second example, the focus of the student's reasoning is on the concept of 'normal rings' to which he then draws comparisons to normal subgroups. Thus, in this second example, the student is foregrounding the *target* domain of rings.

Characterizing Activity Using the ARM Framework

Through reintegration with the widely identified dimensions of mapping activity and similarity, the dimensions of intra-domain activity and attending to differences, and the component of foregrounding domains offer insights into detailing ways in which student mathematical activity during analogical reasoning can be characterized and interpreted. In this section, I use the ARM framework to characterize several identified mathematical activities during analogical reasoning. Table 1 provides a brief overview of the activities characterized and described in this section.

Table 1: Overview of Identified Activities

Activity	Description of Activity	Dimensions of Activity
Exporting	Projecting known aspects of the source domain into the target domain.	• Inter-domain • Attending to similarities • Foregrounding source
Importing	Selectively pulling aspects of the source domain into the target domain.	• Inter-domain • Attending to similarities • Foregrounding target
Recalling	Recalling information that one possesses about the source domain.	• Intra-domain • Neither similarities or differences • Foregrounding source
Distinguishing	Recognizing an anomaly between the source and target domain.	• Inter-domain • Attending to differences • Foregrounding source
Adapting	Modifying the target to accommodate a distinction between the source and target.	• Inter-domain • Attending to differences • Foregrounding target

Exporting and Importing Across Domains. The activity of exporting across domains occurs when a student projects an aspect of the source domain over to the target domain. In contrast, importing occurs when a student selectively pulls aspects over from the source domain into the target domain. The terms "exporting" and "importing" are chosen purposefully to be analogous to the meaning of the words in the context of international trade in the sense that you export outward from the country in which you reside, but import into the country in which you reside.

Consider the following example of a student exporting attributes from the source domain of groups to the target domain of rings:

> So, normal subgroup… First condition is that H is a subgroup of G, and the second condition is that gHg^{-1} is a part of H, and then you can say, therefore, H is normal to G. So, now we're going to call this normal subring. We give this one a name. First condition is that S is a subring of R. Second condition, I don't know. Maybe we say rSr^{-1} is in S, just to copy it.

A visual of this student's work is seen in Figure 4 below. In this example, the student is constructing a definition for what they call "normal subrings." The student constructs a "normal subring" by copying over known aspects of normal subgroups into the context of ring theory.

To contrast with the activity of exporting, consider the following example from the pilot study in which a student is making a conjecture about what comes next in the study of ring theory after having developed the concept of subring:

> Like Abelian rings, or like... giving them that type of thing where you give them special names… Special types of rings, "This is the golden ring." So these, you gave me these properties on the last page. But I'm sure if you have all these properties, it's probably a special type of ring.

In this example, the student is importing the objects of "special" groups, such as Abelian groups, from the source domain into the target domain. The emphasis is on the target which is evident by the fact that the student does not immediately assume that there are Abelian rings just as there are Abelian groups. Rather, this student has discriminately chosen which aspect of group theory they wished to pull over into the target domain of rings.

Characterizing the activities of exporting and importing with respect to the components of mapping and comparing, it is clear that each of these activities are examples of inter-domain activity focused on attending to similarity. They are indistinguishable by examining these two components alone. However, a distinguishing feature can be determined by examining which domain is being foregrounded while engaging with the activity.

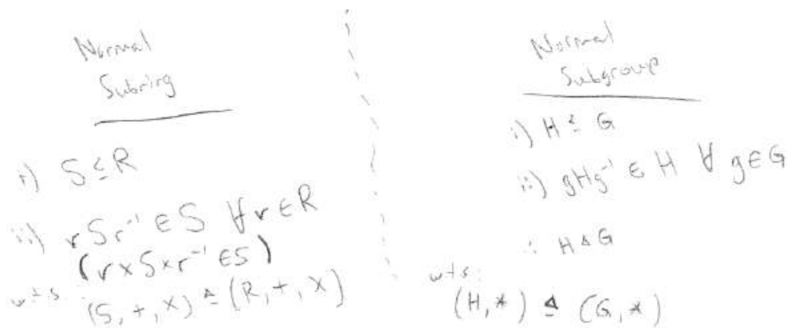

Figure 4: Student Exporting Properties from Source to Target

Recalling Source Knowledge. By expanding the component of mapping across to domains to include intra-domain activity, a greater range of mathematical activities during analogical reasoning become observable. One such activity is that of *recalling source knowledge*. Consider the following example of a student recalling attributes after being asked to make a conjecture for a structure in ring theory analogous to group homomorphisms:

> So, let me just try to recall that… So, group homomorphism. So, there exists a phi that maps from *A* to *B*. So, *A* ... or, I guess maybe it'd be easier to say phi maps from $(A,*)$ to $(B,*)$. So, phi of *a* equals some *b*.

In this case, the student is foregrounding the source and is neither attending to similarity or difference. Although the student is not explicitly engaging in reasoning by analogy in this example, the student is recalling information about the source domain with the intent of utilizing the information for the purpose of analogical reasoning. This is evidenced in the following quote from the same student in which the student exports from the source:

> So, now let's talk about ring homomorphism. Homomorphism. I don't know, I feel like it would be the same thing. There exists, let's just say psi or something from one ring to another ring such that psi is one-to-one. Or, I don't know what you say, onto.

Distinguishing and Adapting. Just as expanding the component of mapping to include intra-domain activity revealed new perspectives on analogical activity, so too does adding the dimension of attending to differences along with similarities provide new insight. One such activity is *distinguishing* between domains. Distinguishing occurs when a student recognizes an anomaly between the source and target domain. Consider the following example in which a student is making observations about the definition of ring:

> So it has ... what I noticed immediately is that, it has two binary operations addition and multiplication. Whereas with group theory we're only dealing with one binary operation at a time.

It is clear from this example that the student is attending to differences and mapping objects between the source and target domains. During distinguishing activity, the student is foregrounding the source. This is due to the nature of anomalies being detected within the target only by a direct comparison to what is already known about the source.

The attention to differences in examples such as this impacted the student's reasoning by analogy later on in the interview process through the activity of *adapting the target domain*. Adapting occurs when a student modifies the target to accommodate a distinction between the source and target. Consider the following statement from the student as they conjectured about the definition of ring homomorphism:

> What would be one for.... We have two [operations] here. Start with phi going from G to H. There's two operations here so I'm like, I don't exactly know if it should just be one of them, or both of them, or how I would do that here. Could I do like three elements, like *a*, *b*, and *c*, and then have like the addition and multiplication?

A visual of this student's work is seen in Figure 5. In this instance, the student is keying in on the difference she identified between the domains two interviews prior and attempting to adapt the homomorphism property she learned in group theory to the context of rings. Adapting activity is characterized by inter-domain activity and attending to differences. Unlike distinguishing activity, adapting activity foregrounds the target since the focus is on constructing new information within the target.

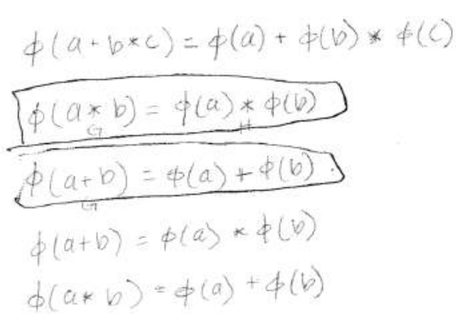

Figure 5: Student Adapting a Structure to the Target

Discussion

An Application to a K-12 Context

Although the ARM framework was developed in the context of abstract algebra, I argue the framework has utility across mathematical contexts. I use an example of students' reasoning about integer operations to illustrate the utility of the framework to capture analogical activity in other contexts. Consider the following statement in which a child is solving the problem -5 – (-3) = ☐ (Bishop et al, 2014).

> Five minus 3 is an easy fact for me. So, um, using negatives it will probably be the same thing like using normal numbers. It will probably be the same thing, but with negatives it'll probably be negative 2.

The child is mapping between the source domain of "normal" numbers and the target domain of negative integers. Because they are emphasizing their knowledge about how subtraction of positives work, they are foregrounding the source. In addition, it is clear that the child is attending to similarities (i.e., "…it will probably be the same thing…") This characterization of the child's activity allows us to see that the child is engaging in the activity of *exporting* from the source domain. To summarize this brief analysis, this child appears to be relying heavily upon their knowledge about "normal numbers" in order to understand a subtraction problem with negative numbers.

Conclusions and Implications

Analogical reasoning in mathematics provides students with the opportunity to develop their conceptual understanding in mathematics that is rich in connections across mathematical contexts (Sidney & Alibali, 2015). Within this study, students were provided the opportunity to leverage analogy and analogical reasoning to make comparisons across domains and reinvent mathematical structures by analogy. By characterizing mathematical analogizing activity, we can support students in coming to create connections across mathematical domains by providing a tool for carefully analyzing how students engage with analogical reasoning in mathematics.

For the purposes of research, the ARM framework provides a foundation for analyzing student analogical activity specific to mathematics. Thus, the framework can be used for generating insight into student's thinking involving analogical reasoning. The results in this paper have only shown a snapshot of students' analogical reasoning in an interview setting. Further research can focus on how student's leverage analogical reasoning over an extended period of time. Finally, the ARM framework and the analogical activities identified in this paper can aid instructors in developing tasks which leverage analogical reasoning by attending to the possible activities of their students while working the tasks. Future research should address ways in which to foster more productive reasoning by analogy in the context of instruction.

References

Bishop, J. P., Lamb, L. L., Philipp, R. A., Whitacre, I., Schappelle, B. P., & Lewis, M. L. (2014). Obstacles and affordances for integer reasoning: An analysis of children's thinking and the history of mathematics. *Journal for Research in Mathematics Education, 45(1)*, 19-61.

Corbin, J., & Strauss, A. (2015). Basics of qualitative research: Techniques and procedures for developing grounded theory. Los Angeles: SAGE.

Crippa, D. (2017). Descartes on the unification of arithmetic, algebra and geometry via the theory of proportions. *Revista Portuguesa de Filosofia, 73(3/4)*, 1239-1258.

Gentner, D. (1983). Structure-mapping: A theoretical framework for analogy. *Cognitive Science, 7(2),* 155-170.

Holyoak, K. J., & Koh, K. (1987). Surface and structural similarity in analogical transfer. *Memory & Cognition, 15(4)*, 332-340.

Holyoak, K. J., & Thagard, P. (1989). Analogical mapping by constraint satisfaction. *Cognitive Science, 13(3)*, 295-355.

Sidney, P. G., & Alibali, M. W. (2015). Making connections in math: activating a prior knowledge analogue matters for learning. *Journal of Cognition and Development, 16(1)*, 160-185.