# HOW MATHEMATICAL MODELING ENABLES LEARNING?

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*In this theoretical paper we compare the Piagetian perspective on knowledge construction to mathematical model construction, with the aim to understand how mathematical modeling enables learning of mathematics and learning of science, as is often claimed. We do this by examining data through two lenses:(i) examining the role of cognitive conflict as it arises during validation of a model and (ii) viewing model validation as a reflection on activity-effect relationship. We explain why we chose to look deeply into model validation specifically, present examples for each lens, and consider implications.*

Keywords: Learning Theory, Modeling, Cognition

There has been much interest over the past few decades in the teaching and learning of mathematical modeling. Typically, investigations seek to understand the process of model construction. However, research has also looked into how learning of curricular mathematics beyond modeling may occur as students generate and validate their mathematical models (Zbiek & Connor, 2006). Taking on a Neo-Vygotskian, socio-cultural perspective, Zbiek and Connor elaborated on the cognitive processes that constitute modeling as to situate thinking about how learning takes place during mathematical modeling. In addition, empirical studies have also shown how a modeling approach to instruction may have an impact on student achievement (e.g. Czocher, 2017; Schukajlow et al, 2012). At the same time, two lines of inquiry have used mathematical modeling as an instructional paradigm to guide students' construction of mathematical knowledge. The first uses mathematical modeling tasks to teach mathematical concepts (Lesh et al., 2000) and the second uses the term model to capture the evolution of conceptual models through mathematical activity (Gravemeijer, 1999; Lesh, Doer, Carmona, & Hjlmarson, 2003). Both lines of inquiry agree that mathematics can be learned through constructing models. However, for one to know how mathematical modeling can best be leveraged to learn mathematical concepts, one first needs to understand how mathematical modeling may enable learning. In this paper, we illuminate data drawn from cognitive modeling task-based interviews using two theoretical lenses on mathematical modeling in order to elaborate how learning may enabled through mathematical modeling.

## Perspective on Learning and Knowledge Construction

In order to understand how learning is occasioned through modeling we take on a Piagetian view on learning and knowledge construction. In this view, learning is considered as a process of transforming one's way of knowing and acting. According to Piaget, all construction consists of activity and all activity is goal-directed. In this sense, all construction (of cognitive structure) is goal directed (von Glasersfeld, 1983). Hence, we begin from the position that mathematical modeling is a goal-directed activity and the modeler is working towards an anticipated model as a goal. Two theories have been highlighted in the constructivist perspective as ways of learning to occur: the theory of equilibration and reflective abstraction. To support our view of modeling as a process of construction, we adapt both these views to mathematical modeling and compare their merits.

### Theory of equilibration

One tenet highlighted in constructivist theory is that conceptual transformation is induced by a perturbing experience. *Perturbation* is experienced when the cognizing subject is met with a constraint or clash in the externalized world and therefore goes through adaptation to regain

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equilibrium (absence of clashes). According to Piaget, disequilibria is stimulated by conflict, either between an individual's action schemes and external realities or among different schemes within an individual. The cognitive structure undergoes assimilation and accommodation repeatedly until it seems "fit" in the externalized world. A *scheme* is an intellectual structure that organize events as they are perceived and classified according to common characteristics. *Assimilation* is the cognitive process by which a person integrates new matter into existing schemata or patterns of behavior. Assimilation does not result in a change of schemata, but it does affect the growth and its part of the development. *Accommodation* modifies the cognitive structure (scheme) to make it "fit" the external world. According to Piaget, accommodation can happen in two ways: one can create a new scheme in which to place the new stimulus or modify an existing schema so that the stimulus fits into. Both forms of accommodation result in change in the configuration. Piaget refers to the process of assimilation and accommodation as *adaptation* (Wadsworth, 2004).

Scholars have since explored the contours of disequilibria and cognitive conflict in different ways. Limon (2001) defined cognitive conflict as something that occurs when a students' mental balance is disturbed by experiences that do not fit their current understanding. Zazkis & Chernoff (2007) stated cognitive conflict is "invoked when a learner is faced with a contradiction or inconsistency of his or her ideas" (p. 196). Berlyne (1970) elaborated cognitive conflict as "a condition in which mutually interfering processes occur simultaneously and in which selection of a motor response from a set of competing alternatives is therefore hampered" (p. 968), which is more amenable to empirical work seeking to understand it in the context of mathematics teaching and learning. Zazlavasky (2015) argued that perplexity, confusion and doubt are often associated with and evoked by cognitive conflict, suggesting that they may be used as proxies for identifying instances of cognitive conflict. Within the literature on mathematical modeling, Lesh et al (2003) identified three kinds of cognitive conflicts arise as models are constructed: within-model mismatches, model-reality mismatches, and between-model mismatches. Researchers have studied how cognitive conflict influences or changes a students' conceptual understanding (Chan, Burtis, & Bereiter, 1997; Ernest, 1996). At the same time, there is also a body of research questioning the role of cognitive conflict in the learning of a concept with evidence that cognitive conflict is only one of the many important factors contributing to learning a concept (Kang, et al., 2004; Zimmerman & Bloom, 1983).

## Theory of Reflective Abstraction

The theory of equilibration only considers how a conceptual change is established when there is a presence of clashes between the cognizing subject and the stimuli. However, it is incapable of explaining how we learn during the absence of clashes. Reflective abstraction addresses this issue. Piaget's (2001) reflective abstraction is a process by which higher level mental structures could be developed from lower level structures. This is done in two phases. In the first phase, the structure at the lower developmental level is projected onto a higher level and in the second phase these structures are reorganized (Campbell,2001). Piaget (2001) acknowledged that reflective abstraction is not necessarily a conscious process.

Reflective abstraction was a significant contribution to addressing the learning paradox (Pascual-Leone, 1976) because it allows for knowledge to be constructed from already-existing knowledge. Simon and colleagues elaborated on reflective abstraction, offering a new explanation for conceptual learning in mathematics that not only addresses the learning paradox but also can contribute to the basis for the design of mathematics instruction (Simon, Tzur, Heinz, & Kinzel, 2004). The mechanism, Reflection on Activity-Effect Relationship (Ref*AER) builds on von Glaserfeld's (1995) tripartite model of a scheme: (1) recognition of a certain situation (S), (2) specific activity associated with that situation (A), and (3) the expectation that the activity produces a certain, previously experienced result or the anticipated the activity-effect relationship (A/E) (Tzur & Simon, 2004). According to Simon and colleagues, an occasion that can result in learning is present when a learner sets a goal (G). The goal is then assimilated into situations (S) that are part of the learner's existing conceptions. From the set of conceptions related to S, activities (A) are called upon to work towards the goal to which the learner anticipates the effect of these activities (A/E). While carrying out these activities, the learners' mental systems engage in continual monitoring, including distinguishing effects of the activity that advance the goals from effects that do not advance them. During the reflection, the learner identifies patterns in the outcomes and abstracts a relationship between the activity and the effect it had on reaching the goal. This abstraction results in a new activity-effect relationship. Here, activities refer to mental activities, the learners' goal are not necessarily conscious, and the effects are the assimilatory conceptions that the learner brings to the situation.

## Perspective on Mathematical Model Construction

We view mathematical modeling as a goal-directed activity. To elaborate the modeling process, we appeal to the cognitive perspective on modeling (Kaiser, 2017) where a mathematical model is considered to be a cyclic process that transforms a real-world problem into a mathematical problem. From this perspective it is common to represent model construction through a mathematical modeling cycle (MMC) such as Blum & Leiß's (2007) characterization. Empirical studies have described dimensions along which a model can change as it is constructed (Czocher & Hardison, 2019) and different ways a modeler can validate her model (Czocher, 2018). Validation is a crucial part of mathematical modeling, because non-viable models are of little use for solving real-world problems. In many mathematical modeling cycles, validating occurs at the end of the process (e.g. Blum & Leiß's, 2007). However, Czocher (2018) argued that validating not only occurs when one checks the final results against the real-world phenomena, she attempted to model but in different ways throughout model construction. When a student attempts to validate her model, she holds two models in her mind: the model she is constructing and the model she anticipates constructing. As a consequence of this comparison, the modeler chooses to accept, revise, or reject the model she is constructing. In this way, validating is responsible for the iterative nature of modeling as well as ongoing monitoring (Czocher, 2018). Therefore, we conclude that since (a) the outcomes of validating lead to modifications of the model, and (b) modelers validate both their final products and monitor their evolving models, validating has a significant contribution in model construction.

For these reasons, we argue that looking deeply into model validation will lead us to understand how learning happens through modeling. To move the field forward, the paper focuses on what happens during validation that leads to the acceptance, rejection or revision of the model, specifically by looking at model validation through two related but different lenses: (1) cognitive conflicts during model validation and (2) viewing validation as a ref*AER. Informed by the review of the relevant constructs, we conceptualize cognitive conflict that arises during validating the model as a discomfort the modeler experiences due to a perceived discrepancy between the model under construction and the model she anticipates constructing. At the same time, validating can be seen as a reflection on Activity-Effect relationship. When a student engages in a modeling task, she is working towards a goal(G) of modeling a real-world situation. To reach this goal, she calls upon activities or activity sequences (A), which she had previously abstracted as having certain effects (A/E), that will help her to map her understanding of the real-world situation to a mathematical structure. While executing these activities, she then monitors the effects of these activities through the interpretation of her constructed model. Then, validation is the reflection that compares the anticipated effect to the constructed effect. As Simon and colleagues stated, "the ability to set the goal subsumes the ability judge the results" (2004, p.318).

We make the case that if cognitive conflicts and reflective abstraction contribute to the construction of knowledge, then in the mathematical modeling context, it is through model validation that cognitive conflict and ref*AER enable learning. This paper first presents an analysis using the first lens to investigate the decisions made during validation, addresses the constraints, and then presents the second lens that could address the limitations of the first.

## Methods

Data for this study were drawn from a larger study of one-on-one modeling task-based interviews with undergraduate STEM majors at a large university in the United States. The students were enrolled in a semester course on differential equations. The overarching goal of the interviews were to explore and document students' mathematical reasoning during modeling. We present examples, to illustrate our case, from one student Jayden, working on the falling body problem.

> The falling body problem: On November 20, 2011, Willie Harris, 42, a man living on the west side of Austin TX died from injuries sustained after jumping from a second-floor window to escape a fire at his home. What was his impact speed?

Jayden was purposefully selected to look deeply into the mechanisms of model validation, because he employed multiple strategies to model the scenario and exhibited observable modeling mechanisms that helped us in explaining our lenses on model validation. Our primary research goal was to build second-order models (Steffe & Thompson, 2000) of his mental activities to explain the factors that shaped his decisions about revising his mathematical model (or not) as an outcome of his engagement in model validation. Since we did not have direct access to Jayden's mental activities, the second-order models are what we inferred from Jayden's observable activities including his language, verbal descriptions and discourse, written work, and on occasion gestures, when they were salient.

For our retrospective analysis of Jayden's engagement with the falling body task, we carried out five rounds of data analysis to arrive at examples that could serve for theory-building. First, we coded the interview for instances of validating, using the method of constant comparison and according to the operationalization in Czocher (2018). Next, we surveyed the validating instances for any identifiable cognitive conflicts and these instances were isolated. Third, we selected examples illustrating cognitive conflict to seek evidence of learning. Fourth, we catalogued instances of validation that failed to be instances of cognitive conflict. Fifth, we applied ref*AER to explain the failed examples. Below, we share illustrations of the third and fifth steps.

## Findings

### Lens 1 - Cognitive Conflicts during Model Validation

We offer two illustrations of when cognitive conflict arose for Jayden during model validation. We leverage the illustrations to explain how Jayden modified the model under construction to accommodate the anticipated model or otherwise left the conflict unresolved.

Jayden began from kinematics equations and successfully modeled the falling body situation without accounting for air resistance. He justified his choice, asserting that air resistance would be negligible "when there is either no air or no fluid to fall through, or you were infinitely close to the ground." The interviewer challenged Jayden to take air resistance into consideration. In response, he constructed a first order, linear, homogeneous equation to model the falling body. He wrote $\frac{dQ(t)}{dt} + \beta Q = 0,$ where Q represented the position of the body and $\frac{dQ}{dt}$ represented its velocity. He then wrote the generic solution $Q(t) = Ce^{-\beta t}$. Jayden wrote $Q(t) = Ce^{-\lambda t}$ with the intention of figuring out "what λ has to equal". Jayden modeled the situation with the initial condition for position as $Q(0) = 0.$ Later, Jayden indicated that he was not sure if the model he constructed was correct. Jayden stated, "I'm not sure if that's right, I'm not sure if there should be some sort of constant increase as you get faster". He drew two graphs showing an increasing relationship between velocity and the air resistance (figure 1). However, he was unsure which representation best matched the situation. He indicated that the linear relationship or the exponential relationship will determine if $\frac{dQ(t)}{dt} + \beta Q$ would equal zero or would equal a forcing term, respectively.

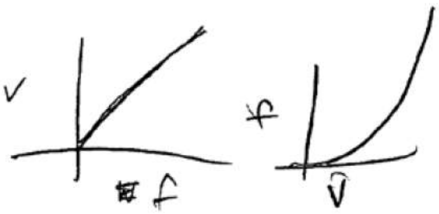

**Figure 1: Students' representation of the relationships between drag force and velocity**

He continued to solve the differential equation assuming the initial positions and initial velocity to be zero. He substituted the general solution $Q(t) = Ce^{-\beta t}$ in $\frac{dQ(t)}{dt} + \beta Q = 0$ and obtained the expression $\lambda Ce^0 + \beta Ce^0 = 0$ which resulted in$\lambda = -\beta$ (figure 2). He then engaged in validating the model he presented by commenting on the reasonableness of it by stating the following:

> it doesn't really tell me a whole lot because I don't know what the graph should look like. I feel like it probably equal some sort of forcing term…because I don't think that the solution would end up being…as he increases in position, I don't think it's going to be $Ce^{-\beta t}$ ... I don't think that this correctly models it.

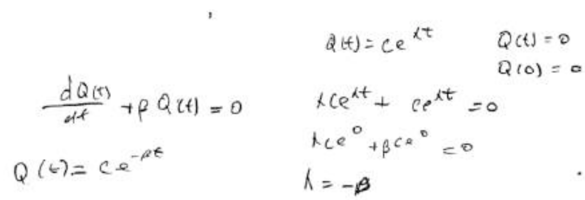

**Figure 2: First order linear differential equation with initial conditions**

Jayden engaged in model validation when he commented on the reasonableness of the model. Here, the model under construction is the mathematical expression based on the assumption that the velocity and force change linearly and the anticipated model is the mathematical expression based on his assumption that "as the velocity gets larger, the force might get greater and greater and greater" . Jayden was experiencing a conflict between the model he constructed and the model he idealized, hence anticipated.

Jayden was able to resolve the conflict when he realized that "the wind is always just an opposing force [so] it could be treated like the force of friction." He then rejected his mathematical model by attempting a different solution that used Newton's laws of motion because they incorporated the surface area of the body and air resistance. In this episode, Jayden attended to the model under construction by modifying the assumptions that the model was based on in order to accommodate the anticipated model. We inferred, based on his sketches, that his anticipated model was his idealization (based on his real-world knowledge) that as the velocity increases the force due to air resistance should increase nonlinearly.

Next is an example where Jayden left the conflict unresolved. Assuming the presence of air resistance, Jayden modelled the falling body using Newton's laws of motion, taking into the consideration the surface area of the falling body and a coefficient to capture the influence of air resistance. He introduced the downward force that the body would experience as $F = ma,$ the air resistance as $F_{wr} = \mu_w \cdot s_a$, and the net force the body would experience as the addition of the two forces. Here $\mu_w$ is was the coefficient of air resistance and $s_a$ was the surface area (Figure 2). However, he mentioned that the velocity should be somewhere in these equations as well. This was evidenced by the following statement he made:

> I just kind of thought of something. His velocity should be somewhere in here also. Because the faster you go the more the force will be…but I have no clue how to put that in.

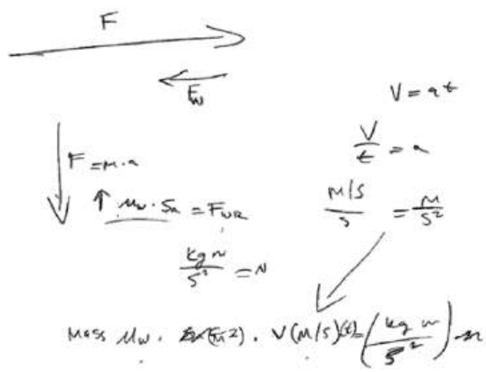


**Figure 3: Student's model of the falling body including air resistance and surface area.**

In order to incorporate velocity in his model Jayden performed a dimensional analysis to balance both sides of the equation in terms of units. He equated $1N = \frac{kg \cdot m}{s^2}$, to the units of $\mu_w \cdot s_a$. While performing the dimensional analysis, he decided that the surface area should not be there. He scratched out the symbol for surface area and instead added the "change in velocity for a time" of the body to the expression (Figure 3). He equated the mass of the body times the coefficient of air resistance times the "change in velocity for a time" of the body to net the force that the body would experience due to air resistance. After arriving at the aforementioned model, Jayden explained:

> Intuitively I don't think I trust that…I mean that's the answer that I reached, but I really think that has something to do with the surface area. Because this pencil will drop faster [drops his pencil from his hands] than a big piece of paper weighing the same amount…so I don't know.

In this instance, Jayden validated the model by commenting on the reasonableness of it, appealing to his lived experiences. Jayden's statement that the model was not trustworthy indicates that he experienced a cognitive conflict. In this case the model under construction was the mathematical expression he produced (without surface area) and the anticipated model was his idealized view of the world, where an object's surface areas affects its velocity through air resistance. Jayden indicated that he did not know how to rectify the dispute and therefore presented this as the final expression for wind resistance. He then discussed how he would set the force equal to $\frac{dQ(t)}{dt} + \beta Q$, obtained from earlier work, in order to find the falling man's impact speed. In this scenario, Jayden accepted his model. However, the conflict was left unresolved.

While analyzing cognitive conflicts during model validation was a useful way to look at what happened during model validation that led to the acceptance, rejection, and revisions of the model, there were limitations to it. First, taking this perspective assumes that learning during mathematical modeling only occurs during the rejection or/and revision of the model. This is not necessarily true. Learning could also happen when one is satisfied with the model and accepts it because accepting the model may also have transformed the modelers way of knowing and acting about the model. This perspective ignores this case. Second, not all validating instances coincide with instances of cognitive conflict. Therefore, it is necessary to explain such instances where model validation is present, but conflict is not. The second lens of looking at validating was drawn upon to address some of these limitations.

## Lens 2 – Validation as a Ref*AER

The following is an example of model validation which could not be explained through the first lens, can now be explained by viewing validating as a Ref*AER. Recall the scenario where Jayden modelled the falling body with air resistance with the expression $\frac{dQ(t)}{dt} + \beta Q = 0$ and initial values $Q(0) = 0$. While considering the initial conditions to solve the differential equation, Jayden stated:

> I'm just trying to think what initial conditions I need to use. I guess I'll have to just say… $Q(0) = 0$ because his position is 0. But I guess it will be better if I said that this was [pause] let's see [long pause] … I guess this is fine [pointing at $Q(0) = 0$].

Jayden validated his model through evaluating the reasonableness of the initial condition $Q(0) = 0$. However, he was not experiencing a conflict because there was no evidence for a discrepancy between the model under construction and the anticipated model. When Jayden stated "I'm just trying to think what initial conditions I need to use" we take that as an indication of him recalling the activities that would lead him to the desired effect and filtering the ones that would not. Here the goal is to solve the differential equation (G), the activity is drawing on the appropriate initial condition (A), and the effect is what comes out of solving the differential equation using the selected initial condition (E). Jayden first considered the initial condition $Q(0) = 0$, and next he considered whether they would advance him toward his desired goal. This is evident when he said, "But I guess it will be better if I said that this was…" Through reflecting, Jayden ultimately conformed to his initial choice $Q(0) = 0$, and therefore accepted his model. In this instance, Jayden was continuously monitoring and reflecting on the effect of selecting $Q(0) = 0$ as the initial condition would have towards reaching his ultimate goal.

The following is an example where Jayden rejected his model, which can also be explained using the Ref*AER lens. Jayden's initial approach was to draw from the equations of motion from mechanics. To find the impact speed of the falling body, Jayden wrote the equation $s = ut + \frac{1}{2}at^2$, where $s$ is the distance the body travelled, $u$ is the initial velocity, $a$ is the acceleration due to gravity, and $t$ is the time it took to travel a distance $s$. As soon as he realized that the equation contains the time of fall $t$, Jayden scratched out the expression and resorted to $v^2 - u^2 = 2as$. The reason being the first expression required the time of fall, which was not given in the task. This was an instance of validation because he scratched out the first expression and attempted a different solution. However, there was no evidence of conflict. In this instance, the goal for Jayden was to find the impact speed without using the time of fall (G). He stated, "I could find the time of fall, but it's not necessary". His activity (A) was selecting $v^2 - u^2 = 2as$ over $s = ut + \frac{1}{2}at^2$ through cataloguing existing equations and reflecting on the effect they had in reaching the desired outcome (E). As a result of validating, he rejected his initial expression and selected another one to meet his desired effect.

## Discussion & Conclusions

This study investigated the mechanisms of model validation through two lenses: (i) looking at cognitive conflicts that arise due to the discrepancy between the model under construction and the anticipated model, and (ii) viewing model validation as a reflection on activity-effect relationship. Our analysis offers insight into potential mechanisms for model construction and suggests a strong link between model construction and Piagetian explanations of knowledge construction. Studying the nature of cognitive conflicts students experience while engaging in mathematical modeling and viewing model validation as Ref*AER may be an avenue towards elaborating how learning occurs through mathematical modeling because it may inform us about how students make decisions about the viability of their models.

Given the preceding analysis, we close with two considerations: limitations and future directions. This study only informs us how learning may be enabled through mathematical modeling and is not capable to inform us on what was learned. At the same time, the paper does not discuss the explicit treatment of the two lenses and how they can be leveraged to analyze the mechanism of model construction, yet. Future analysis will investigate this. In order to understand what was learned through modeling, instances of validating will be analyzed closely, using the lenses presented in this paper, to see the following: why do modelers chose to accept, revise, and reject the models? how do they do so? and in what ways? However, this theoretical paper outlines the extent to which these learning theories are applicable to mathematical modeling. This we believe is a significant contribution as it sets us open to understanding what is it that is being learned through mathematical modeling. These mechanisms can then be leveraged to develop instructional theory that fosters mathematical conceptual learning through mathematical modeling.

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