FRAMEWORK FOR REPRESENTING A MULTIPLICATIVE OBJECT IN THE CONTEXT OF GRAPHING

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In this study, based on the analysis of a teaching experiment with middle school students, we propose a framework for describing meanings of a point represented on a plane in terms of multiplicative objects in the context of graphing. We classify those meanings as representing (i) non-multiplicative objects, (ii) quantitative multiplicative objects (Type-1 and Type 2), and (iii) spatial multiplicative objects. We then discuss implications of these meanings with respect to students' graphing activities.

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Quantitative and covariational reasoning play a critical role in students' understanding of various ideas in mathematics (e.g., Carlson, Jacobs, Coe, Larsen, & Hsu, 2002; Ellis, 2011; Johnson, 2015). There are numerous aspects and important constructs (e.g., quantitative structure, frames of reference, and quantification) that individuals could develop in order to engage in productive, powerful quantitative and covariational reasoning (Joshua, Musgrave, Hatfield, & Thompson, 2015; Moore, Liang, Tasova, Stevens, 2019; Thompson, 2011). One of these constructs is a *multiplicative object*. Thompson, Hatfield, Yoon, Joshua, and Byerley (2017) suggested that students "must construct a multiplicative object of quantities' attributes in order to reason about their values covarying smoothly and continuously" (p. 128). Constructing a multiplicative object is also important in the context of graphing as it imparts a productive meaning to a point on a graph (Frank, 2017; Saldanha & Thompson, 1998; Thompson & Carlson, 2017). However, much is left to understand about the extent and nature of students' meanings of a point as representing a multiplicative object in the context of graphing. Therefore, in this study and against the backdrop of empirical data, we provide a framework to classify students' meanings of a point on a plane in terms of representing a multiplicative object.

What is a Multiplicative Object?

A multiplicative object can be considered a conceptual object that is formed by uniting in the mind two or more quantities' magnitudes or values simultaneously (Saldanha & Thompson, 1998; Thompson, 2011; Thompson & Carlson, 2017). The mental operation of someone who constructs a multiplicative object is similar to the operation of someone who conceives a quarter coin as being, simultaneously, a circle and silver in color. In this operation, circle and silver, as two attributes of the object, have been considered, simultaneously, as one property of a quarter coin. For a dynamic example, imagine heating the quarter coin up to the melting point of silver. As a multiplicative object, coupling the two attributes, someone could track the variation of the coin's color with the immediate and persistent awareness that, at every moment, the temperature of the coin also varies.

In the context of co-variation, Thompson (2011) represented the multiplicative object formed by uniting two quantities' variations by using the following representation: $(x_e, y_e) = (x(t_e), y(t_e))$, where $x_e = x(t_e)$ represents a variation in the values of x, where t_e represents variation in t through conceptual time over the interval $[t, t+e)$. He explained that in order for students to reason covariationally, they must unite x_e and y_e by constructing (x_e, y_e), which simultaneously represents the two. Note that the corresponding representation of this conceptual object in graphical context would be a point in a coordinate plane. We next discuss the role of conceiving a point as a multiplicative object in developing meanings for graphs.

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Multiplicative Objects in the Context of Graphing

Despite the notion that graphing is critical for understanding various ideas in STEM fields (Rodriguez, Bain, & Towns, 2019; Kaput, 2008; Leinhardt, Zaslavsky, & Stein, 1990), students face a number of challenges (e.g., graphs as pictorial objects) in interpreting and making sense of graphs (Clement, 1989; Leinhardt et al., 1990; Moore & Thompson, 2015). Thompson and Carlson (2017) conjectured that part of these students' difficulties were grounded in being unable to conceive points on a graph as multiplicative objects, and several researchers have provided evidence to this claim (e.g., Frank, 2016, 2017; Stalvey & Vidakovic, 2015; Stevens & Moore, 2017; Thompson et al., 2017). Given this evidence, conceiving points as multiplicative objects might be an integral part of constructing productive meanings for graphs, and thus a student's construction of points should not be taken for granted.

We note that students' meanings of points on a plane can be considered as a representation of multiplicative object if students conceive a point by engaging in multiplicative operation—the operation of uniting and holding in mind two attributes of an object (i.e., quantities) simultaneously—as defined by Inhelder and Piaget (1964). Inhelder and Piaget first introduced the role of a multiplicative operation to characterize children's thinking when classifying 2-atrribute objects (e.g., objects grouped according to shape and color, as described above). They reported two different ways of children's thinking, both of which lead to normative responses when identifying a missing element in a matrix arrangement (see Figure 1). One is based on twofold symmetries that involve relying on perceptual configuration of the matrix arrangement and treating it as an incomplete pattern. For example, squares are symmetric over the horizontal axis of the diagram, so the blank space should include a circle. Similarly, red objects are symmetric over the vertical axis, so the blank space should include a blue circle. The other way of thinking is based on a multiplicative operation on a logical structure with reasoning about objects and coordinating two classes. For example, classifying the given three objects simultaneously in terms of shape and color, then identifying two elements of squares already belong to the classification of red or blue, noting the given element of circle belongs to red, and then joining circle and blue to construct the missing element.

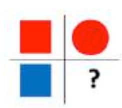


Figure 1. A matrix diagram designed based on the narratives of Inhelder & Piaget (1964).

We rely on these two types (e.g., perceptual features vs. reasoning about attributes) to classify students' meanings of points as representing non-multiplicative objects or representing multiplicative objects. In addition, we note that Inhelder and Piaget (1964) illustrated that an arrangement does not have to be in a matrix form for a child to think of objects in terms of two attributes. Inhelder and Piaget reported students could coordinate multiplicative classes without needing objects in a matrix form, and in our work, we illustrate that students could conceive points as representations of multiplicative objects without needing points represented in a Cartesian coordinate system (see spatial multiplicative object of the following framework). In other words, we considered that representing a multiplicative object is not restricted to plotting a point on a coordinate plane in the normative sense; it is about conceiving a point as a simultaneous representation of the two attributes of the same object.

Role of Multiplicative Objects in Emergent Shape Thinking

Moore and Thompson (2015) introduced the notion of *emergent shape thinking* to describe a person who envisions a graph "*simultaneously* as what is made (a trace) and how it is made (covariation)"

(p. 785). Constructing a graph from this perspective involves (1) representing two quantities' magnitudes and/or values varying on each axis of a coordinate system, (2) creating a point as a representation of a multiplicative object uniting those two quantities' magnitudes or values as a single object, and (3) generating a graph by conceiving the process of a multiplicative object leaving a trace when moving within the plane in ways invariant with the two covarying quantities. As we elaborate on different types of representing multiplicative objects, our framework informs the process in which a student could develop emergent shape thinking, which researchers (e.g., Frank, 2017; Moore, Stevens, Paoletti, Hobson, & Liang, 2019) have shown it is a productive way of thinking about graphs.

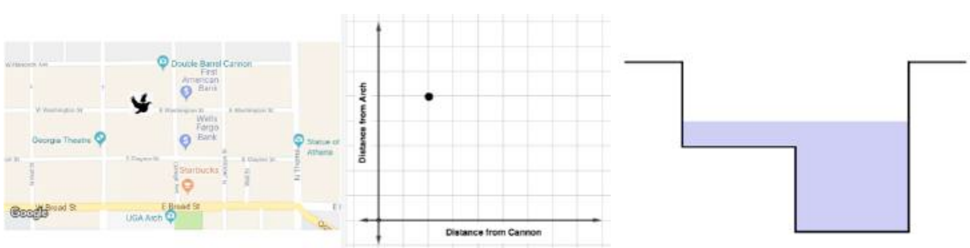
Method

Our work here stems from a semester-long teaching experiment (Steffe & Thompson, 2000) that occurred at a public middle school in the southeast United States. We recruited four seventh-grade students (age 12). Our goal was to investigate students' thinking involved in conceiving and representing various quantitative relationships. In this paper, we focus on two of the four students, Zane and Ella, since their meanings for points were consistent within their representational system and clearly described by them throughout the teaching experiment. We believe it is important to document these ways of thinking in order to add nuances to our models of students' conception of points in a graphing activity in terms of multiplicative objects.

The first author was the teacher-researcher (TR). We recorded all sessions using two video cameras to capture students' work and their gestures and a screen recorder to capture their activities on the tablet device. We transcribed the video and digitized students' written work for on-going and retrospective conceptual analyses (Thompson, 2008). Our analysis relied on generative and axial methods (Corbin & Strauss, 2008), and it was intended to develop working models of students' thinking based on their observable and audible behaviors.

Tasks

Before conducting the teaching experiment, the TR developed an initial sequence of tasks by considering particular design principles focused on graphing covarying quantities (e.g., Frank, 2017; Moore & Thompson, 2015; Stevens, Paoletti, Moore, Liang & Hardison, 2017; Thompson & Carlson, 2017). The TR revised and implemented those tasks based on on-going inferences and analysis of Zane and Ella's thinking. Each task was designed with a dynamic geometry software and displayed on a tablet device.



(a) (b) (c)

Figure 2. (a) The map of Downtown Athens, (b) Coordinate system with a point, and (c) A diagram of the pool

The Crow Task. The situation includes a map of Downtown Athens with a movable crow and fixed seven locations: UGA Arch (hereafter Arch), Double-Barreled Cannon (hereafter Cannon), First American Bank, Georgia Theater, Wells Cargo Bank, Statue of Athena, and Starbucks (see the map in Figure 2a). We also presented a Cartesian coordinate system whose horizontal axis is labeled as distance from Cannon and vertical axis is labeled as distance from Arch (Figure 2b). Students can control the crow freely by dragging it and see how the corresponding point in the coordinate plane changes (go to https://bit.ly/PMENA42 for the digital version of the tasks).

The Swimming Pool Task. This task was adapted from Swan (1985). We presented students a dynamic diagram of a pool (Figure 2c), where they could fill or drain the pool by dragging a point on a given slider. We designed the task to support students in reasoning with the inter-dependence relationship between two continuously co-varying quantities: amount of water (AoW) and depth of water (DoW) in the pool.

Framework for Representing a Multiplicative Object

In this framework, we describe students' meanings for a point in terms of multiplicative objects. We classify those meanings as representing (i) non-multiplicative objects, (ii) quantitative multiplicative objects (Type-1 and Type 2), and (iii) spatial multiplicative objects.

Representing a Non-Multiplicative Object

In this section, we illustrate a characterization of students' meanings for points on a coordinate system as contra-indication of representing a multiplicative object. Building off of limited number of studies (i.e., David, Roh, & Sellers, 2018; Frank, 2016, 2017; Thompson & Carlson, 2017; Thompson et al., 2017), this characterization emerged as we identified students correctly plotting points in the plane by carrying out a certain procedure (e.g., over and up), but the meaning of these points included solely an ordered pair of numbers and/or a location in the plane that did not symbolize or unite two quantities' magnitudes and measures.

Note that there are different students' meanings that could be classified as a non-multiplicative object. Aforementioned researchers have exemplified some of those meanings. For example, Thompson et al. (2017) argued that calculus students viewed the point (2, *f*(2)) in a coordinate plane as a value of the function, instead of the relationship between the value of the function (i.e., *f*(2)) and the value (i.e., 2) for which the function was evaluated. Similarly, David et al. (2018) reported that some students treated the output of the function as the location of the coordinate point in the plane, rather than on the vertical axis (i.e., *location-thinking*). Those students—and consistent with those in Thompson et al.'s study—did not think of 2 as a measure of a magnitude located on the horizontal axis and they did not think of *f*(2) as a measure of a magnitude located on the vertical axis in a canonical Cartesian plane.

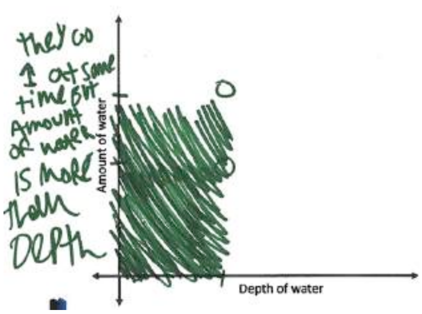


Figure 3. Ella's first draft.

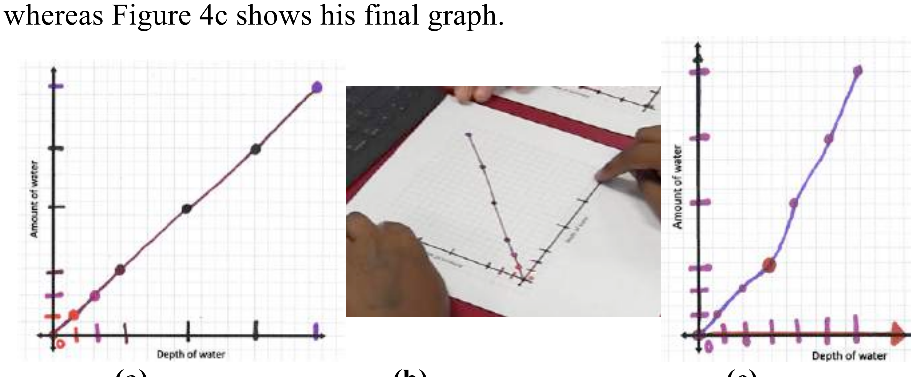
For an empirical example from our data set that falls under this category, we present Ella's graphing activity in Swimming Pool Task. We asked Ella to sketch a draft of a graph that represents the relationship between AoW and DoW as the pool fills up. She began by inserting two tick marks on each axis as an indication of AoW and DoW. As seen in Figure 3, she noted, as we fill the pool up, both tick marks go up along the axis at the same time and wanted to place the tick mark for AoW further along the vertical axis than the tick mark for DoW since she thought "amount of water is more than depth."

Then, Ella drew a small circle in the plane to show "where those two things [*tracing her finger horizontally from the tick mark on the vertical axis to the circle in the plane, then vertically down from the circle in the plane to the tick mark on the horizontal axis*] meet here." Then, she shaded the rectangular area in the plane, what she called "a box" (see Figure 3), to show "a bunch of water." When asked to explain what the small circle meant for her in terms of the pool situation, Ella said, "I don't know what it means" and further she explained "that is just like the dot between [*tracing her finger horizontally from the tick mark on the vertical axis to the circle in the plane*] here [*tracing her finger vertically down from the circle in the plane to the tick mark on the horizontal axis*] so I can just make this box [*pointing to the shaded area in the plane*]." We infer that Ella was able to plot a point in the plane respective of the tick marks that she placed on each axis. However, Ella conceived the point as a landmark to draw "the box," which was a contraindication of *representing* a multiplicative object. Although Ella reasoned about the attributes when placing the two tick marks on the axes and used those tick marks in order to generate the point (i.e., the circle in Figure 3), Ella's meanings of the point didn't include uniting the attributes of an object (i.e., AoW and DoW) in the plane; instead Ella conceived the point in terms of a mark as a part of a procedure to set the corner for the box.

Representing a Quantitative Multiplicative Object (QMO)

In this category, we describe meanings of students who construct and/or interpret a single point in the plane in relation to two quantities whose magnitudes or values represented on each axis. We illustrate this category by using Zane's graphing activity in the Swimming Pool Task.

We asked Zane to sketch a graph that shows the relationship between AoW and DoW as the pool fills up. Zane started with drawing tick marks on each axis. Zane referred to the quantity's magnitude by drawing a line segment from the origin to the tick mark on the axis to articulate his meanings of tick marks. Moreover, Zane simulated the quantities' variation by tracing his fingers along the axis as we played the animation to fill the empty pool (Figure 4b). After inserting tick marks, Zane plotted points for each related tick marks correspondingly (see his color-coded points and tick marks in Figure 4a), then he connected those points with line segments in the plane. Figure 4a shows Zane's earlier graph whereas Figure 4c shows his final graph.



(a) (b) (c)

Figure 4. (a) Zane's draft, (b) Zane moving his fingers on axes, and (d) Zane's final graph

To gain more insights into how he conceived his plotted points, we asked Zane to show the point on his graph representing when the pool is full. Zane first pointed to the far right and top purple tick marks on each axis (see Figure 4a, see also Figure 4b), and he then pointed to the corresponding purple point in the plane (see Figure 4a). Taken together with his description of a dot—"the dot represents both amount of water and depth of water"—his actions suggest that he could associate two tick marks (i.e., indication of quantities' magnitudes for Zane) on each axis to the corresponding point in the plane, which is an indication of representing a QMO.

So far, we demonstrated Zane's meanings of a single point in the plane. We, now, classify the instances of representing QMO in two ways in relation to conceiving a graph (e.g., a line drawn in the plane) when students represent a relationship as two quantities vary: (1) as a path or direction of movement of a dot in the plane, and (2) as a trace of the point consisting of infinitely many points, each of which showing the relationship of two varying quantities. We illustrate those types below.

Type-1 QMO. Type 1 includes students who envision points as a circular *dot* that represents two quantities' magnitudes or values simultaneously and envision that points on a graph (e.g., a line) do not exist until they are physically and visually plotted. Therefore, those students conceive the graph as representing a direction of movement of a dot on a coordinate plane. We illustrate by continuing to discuss Zane's graphing activity identified above.

We asked Zane whether his graph (see Figure 4c) showed every single moment of how the two quantities varied in the situation, Zane claimed no because one would need to stop the animation and plot an additional point in order to show the desired moment and state of the quantities. We infer that, for Zane, his line did not have points until they are visually plotted. He needed to physically plot additional points to represent moments in between two available points, even if there is a line connecting them. When questioned what the line segments that he drew in between dots meant to him, Zane responded that the line shows "where the dots go." By go, he meant a dot moving from one plotted point to the next plotted point, but not in a way that preserved an invariant relationship between those two points.

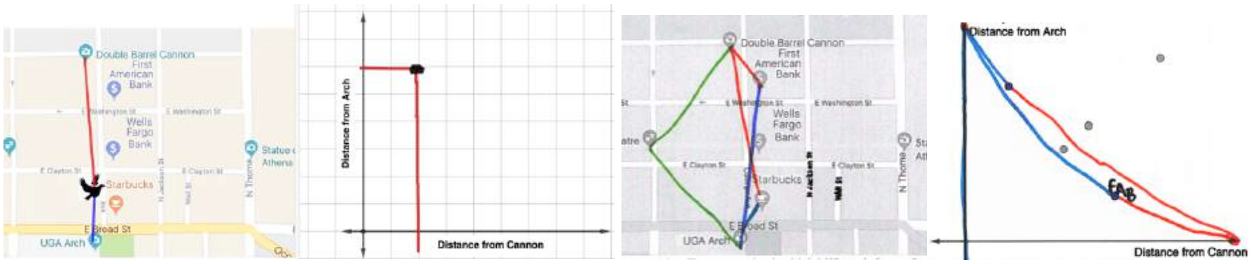
Therefore, despite his success in being able to conceive of a point as a multiplicative object, Zane assimilated his graphing activity as one dot moving along a line path instead of one dot generating infinitely many points by leaving a trace. We claim that his meaning for points and lines played a critical role in Zane's construction and constrained him from conceiving a graph as an emergent, in-progress trace (i.e., the third component of emergent shape thinking).

Type-2 QMO. Type 2 describes students who could envision a point as an abstract object that represents two quantities' magnitudes or values simultaneously, and envision a graph (e.g., a line) as composed of infinitely points, each of which represent two quantities' values or magnitudes, which is an indication of emergent shape thinking. We did not have data in our current study to show an empirical example of a student reasoning emergently (see Moore & Thompson, 2015, for an illustration).

Representing a Spatial Multiplicative Object (SMO)

This category emerged as we coded instances where the students assimilated a "point" in the plane as an object/location by focusing on the object's quantitative properties and engaging in quantitative reasoning (e.g., gross comparison of two quantities' magnitudes). Students who represented a SMO determined the object's *location* by coordinating and representing two (measurable) attributes of the object (e.g., the crow's distance from Arch and Cannon) *in* the plane, as opposed to representing those attributes on the axes of the plane. That is, they represented the two attributes considering the object's (i.e., the point in the plane) distance from each axis or from a certain location in each axis of the coordinate plane.

For an illustration, we present a moment from Zane's activity in a version of the Crow Task. In the previous version, Zane assimilated the given black point in the plane (see Figure 2b) as the crow. In this version, we hided the given point and asked Zane to plot a point that represents the crow's DfA and DfC when the crow is in a place on the map as seen in Figure 5a. Zane began drawing a horizontal line segment starting from the vertical axis to a certain place in the plane and drew a vertical line segment from that place to the horizontal axis (see Figure 5b). Making connection to the blue and red bars appearing on the map (Figure 5a), he referred to the horizontal line segment in the plane saying "the crow's distance form Arch is shorter" and referring to the vertical line segment in the plane, he said "the crow's distance from Cannon is longer." Then, he plotted the black dot (seen in Figure 5b) where these line segments intersected.



(a) (b) (c) (d)

Figure 5. (a) and (b) Zane's activity, (c) and (d) Ella's activity

We infer that to *locate* the black dot (i.e., the crow for him) in the plane, Zane represented the crow's distance from Arch as the distance from the vertical axis and the crow's distance from Cannon as distance from the horizontal axis. Note that this activity is consistent with Zane's earlier activity where he assimilated the axis of the coordinate system as Arch and Cannon. Although the position of the point he plotted is not normative in terms of a canonical quantitative coordinate system (Lee, Hardison, & Paoletti, 2018), this activity was valid for Zane as he was assimilating the black dot in the plane as the crow, and imagining Arch in place of the vertical axis and Cannon in place of the horizontal axis itself.

Note that our emphasis in this framework is not on students' use of coordinate systems (cf. Lee, Hardison, & Paoletti, 2018; Paoletti, Lee, & Hardison, 2018); instead we categorize students' meanings of points in terms of multiplicative objects represented in a space that may or may not be classified as any type of coordinate system that we, as researchers, know. For example, Zane's graphing activity (Figure 5b) may suggest he *seemed* to be engaging in plotting a point as a multiplicative object in a *spatial coordinate system* (according to Paoletti et al., 2018), where Zane established a frame of reference considering the horizontal axis and the vertical axis as a reference point to represent the crow's distance from Cannon and Arch, respectively. However, conceiving a point as SMO should not directly imply representing the relationship on a spatial coordinate system. Considering Ella's activity in Downtown Athens Task (see Figure 5c and 5d), we infer that Ella essentially formed, from our perspective, a two-center bipolar coordinate system based on gross comparisons between the two quantities' magnitudes. That is, she conceived Arch and Cannon as a location on the vertical and horizontal axis, respectively, implied by the labels (see orange dots on each axis in Figure 5d). Then, she made sense of the rest of the space by coordinating the radial distances between "places" in the plane and "Arch" and "Cannon" on each axis. For example, Ella labeled a point as FAB in the plane (see Figure 5d) indicating First American Bank. To justify why FAB, referring to the orange and blue line segments that she drew *in the plane* (Figure 5d), she said "the orange is shorter, and the blue is longer." Referring to the line segments on the map (see Figure 5c), she added, "over here, like the same thing" showing FAB is closer to Cannon and farther from Arch in the map as well as in the plane.

Discussion

In this study, we illustrated different ways students' graphing activity involved multiplicative objects when graphing quantitative relationships. We believe outlining such a framework is important as researchers can be more attentive to those meanings students hold for their representational activity. In this section, we discuss the different implications of these ways of representing multiplicative objects on students' graphing meanings.

QMO VS. SMO

We perceive that the goal for students who conceive a point as a SMO is locating the object in the space, which is why the primary attention is the plane rather than the axes. To locate the object on the space, students represent the quantities' magnitudes in the space by committing to a reference, such as the axis itself or a location on the axis, and where those magnitudes meet determines the location of the object (e.g., the crow, bike, or First American Bank). The meaning of this point on the space is different than students' meanings who conceive a point as QMO as they represent quantities' magnitudes on the axis and form the point by taking two orthogonal magnitudes along the axis and creating projections. In turn, the space is inconsequential beyond creating the point by joining the projections when engaging in representing QMO. For that reason, students who produce a graph by tracing a SMO will perform different actions (e.g., moving different directions on the space) than others who produce a graph by tracing a QMO. Interestingly, those two graphs produced by tracing a SMO and a QMO will be exactly the symmetry of each other over the line of $y = x$ in case of Zane's activity. Despite the fact that students construct non-normative graphs when tracing a SMO, their form of reasoning is productive in terms of completing the goal of the activity as they perceive it. Their activity should be considered as a different way of graphing relationships because it still requires students engage in quantitative coordination in a non-normative way. Calling this type as "spatial" should not imply dismissing the role of quantitative coordination in students' reasoning.

Type 1 and Type 2

To produce a graph via reasoning emergently, students trace a point and anticipate a graph including infinitely many points, each of which is a representation of a multiplicative object; a point as a multiplicative object is sustained throughout conceiving the emergence of the trace. Given the importance of reasoning emergently in developing productive meanings for graphs, by distinguishing two types of representing a QMO (i.e., Type 1 and Type 1), we note that conceiving a point as a multiplicative object is necessary, but not sufficient in envisioning "graphs as composed of points, each of which record the simultaneous state of two quantities that covary continuously" (Saldanha & Thompson, 1998, p. 298). As we illustrated above, students whom we classify engaging in representing Type 1 QMO can conceive a point in terms of multiplicative object, but they do not imagine a trace being produced as representative of that multiplicative object. Given there are numerous students (i.e., about 89% of secondary students [N=1798], as reported in Kerslake, 1981) not conceiving of infinitely many points on a line and believing there is no point on a line until they are plotted (Mansfield, 1985), it becomes important for us to be able to determine which type of multiplicative object the students forming and representing. In doing so, we can inform our instruction to foster and support students in developing productive meanings for graphs.

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