A CONCEPTUAL ANALYSIS OF THE EQUAL SIGN AND EQUATION –THE TRANSFORMATIVE COMPONENT

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Mathematics education scholars have generally classified students' conception of the equal sign as either operational or relational. Adding to these conceptions, Jones (2008) introduced the idea of substitutional conception. Building off these scholars, I introduce a form of understanding the equal sign that includes a transformative equivalence component and extends the conceptions of the equal sign to conceptions of equations.

Keywords: Algebra and Algebraic Thinking, Cognition.

Introduction

Students' algebra achievement acts as a gatekeeper that affects their future academic success and employment (Rech & Harrington 2000; Ladson-Billings, 1998). As a result, students' understanding of algebra continues to attract attention from mathematics educators (Kieran, 1992; National Council of Teachers of Mathematics[NCTM] 2000; Wagner & Kieran, 1989). Researches have established that students' conception of the equal sign is fundamental to their learning of algebra (Knuth et al., 2006; McNeil & Alibali, 2005; Falkner et al., 1999). Consequently, aiding students in building a productive understanding of the equal sign may not only support students learning of algebraic concepts but also foster social equity. Building on such belief, this paper proposes one cognitive model for giving meaning to the equal sign.

The paper begins with a detailed summary of two important papers in which the authors separately addressed elementary school students' understanding of the equal sign and middle school students' understanding of the equal sign (Behr et al., 1980; Knuth et al., 2006). Both studies, although focused on students of different school ages, suggested a differentiation between an "operational" and a "relational" (or "equivalent") understanding of the equal sign, and such differentiation has been echoed by other researchers (Carpenter et al., 1999; Baroody & Ginsburg, 1983; McNeil et al., 2006; McNeil, 2008; McNeil et al.,2011). In general, an operational conception involves interpreting the equal sign as an announcement of the result of an arithmetic calculation, and the relational conception interprets the equal sign as indicating a mathematical equivalence (Knuth et al., 2006). After summarizing these perspectives, I introduce Jones's (2008) notion of the substitutive conception of the equal sign, which further divides the "relational understanding" into "substitutive relational" and "sameness relational." Following this cursory literature review, I explain the theoretical rationale for this paper and introduce a conceptual analysis—named *the transformative model*—along with a brief empirical result. The model extends the question from understanding students' conception of the equal sign to understanding their conception of equation, and it contributes a "*transformative equivalence*" component to previous discussions.

Background

The equal sign was not introduced until 1557 by Recorde, and it was universally applied around 1700. In the field of mathematics, it is not the only symbol that represents an equivalent relationship, and indeed it is a special symbol that only represents a certain category of mathematical equivalence (Molina et al., 2009). Therefore, one can reasonably assume that students' understanding of the equal

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sign might be as varied in both meaning and sophistication as its development across the history of mathematics.

One of the earliest works in studying students' understanding of equal sign can be found in Behr and his colleague's (1980) research, in which the authors studied elementary students' conception of the equal sign. One major finding is that students (around 6-7-year-old) hold a fixed belief that the equal sign has to appear after the operation symbol. For example, some students in their study read the expression "8=5+3" as "five plus three equals eight." Furthermore, students generally rejected a sentence in the form "$\square = 2 + 4$" but instead changed it to "$\square + 2 = 4$" or "$2 + 4 = \square$". One interpretation of them was that students had an inclination in using an "action" sentence rather than a non-action sentence. In such a case, the authors argued that students conceived the equal sign as a "do something signal," and one could only have an equal sign when there was an operation appears on the left (p.16). In other words, students did not conceive the equal sign as suggesting two equivalent expressions but an operation symbol. In the study, some students even changed expression "4=6+1" to "4=6+10" and saying 4 and 6 made 10. The authors further postulated that some students were merely treating equal symbols as symbols to connect numbers.

The aforementioned study provides evidence that many primary school students do not have a flexible way of using the equal sign and frequently see it as an operation signal, and McNeil and Alibali (2005) revealed similar patterns among high school and college students. Following this result, Knuth and his colleagues (2006) conducted a quantitative study on middle school students' conceptions of the equal sign. Based on their findings, they argued that middle school students lack "relational understanding" (or equivalence understanding) of the equal sign, and this influenced students' success in solving algebra problems.

Knuth et al. (2006) used two problems in their study. In the first problem, students were required to give an explicit description of the meaning of the equal sign. In the second problem, students were required to solve algebra problems such as "$4m+10=70$". The authors found that students who explained the equal sign with a relational description (i.e., equal sign means the same as) were more likely to solve the algebra problem correctly than students who explained the equal sign with a non-relational description (i.e., equal sign is a sign connecting the answer to the problem). The authors also illustrated that the students who gave relational descriptions were more likely to use algebraic methods in solving the later problem; the authors defined algebra methods as methods that involved algebraic manipulation (i.e., performing the same transformation on each side of the equation), and non-algebra methods are guessing and trying or direct arithmetic (i.e., 70-10=60, 60÷4=15). The authors further showed that both correlations hold when controlling for students' general mathematic abilities, that relatively few students hold a relational view, and that the percentage of students who hold such view did not increase significantly when students moved from grade 6 to grade 8. The authors suggested such a lack of progress might be related to the lack of explicit focus on the meaning of the equal sign in the curriculum.

The findings from these studies are consistent. In listed studies, we observe students may either understand equal sign as a "do something signal," which aligns with an operational meaning, or as "a sign of equivalent relation," which supports students to use algebraic method in solving it. In both studies, we also observe students with the operational conception have difficulties in doing algebra flexibility. Based on these studies, one can further conjecture that a student, if holds an operational conception, will likely avoid using algebraic methods since an algebraic method requires operations on each expression, but he/she may not want the right-side expression to involve an operation.

Adding to these researches, Jones and his colleague (Jones, 2008; Jones et al., 2012) studied the substitutive conception of the equal sign. They define the substitutive conception as realizing both sides of the equation can be used to substitute each other when needed. These studies are resonated with the study of "relational thinking" developed by Carpenter et al. (2005). In short, the key idea that concerns those researchers is that given “16+37=53”, students should be able to evaluate “16+39” without doing a direct calculation but utilizing the possible connections between these two expressions. More importantly, Jones and his colleague (2012) argued there is a cognitive difference between “seeing two entities as equivalent” and “being able to interchange equivalent expressions when beneficial,” and scholars frequently over-emphasized the sameness component and risks neglecting the substitute component. Jones (2008) found that while making substitutions, students were oblivious to the correctness of the equation. For example, when a student needed to replace 77, some picked 77=11+33. Jones et al. (2012) also observed cases where students realized that both sides of the equation are the same but did not substitute them in problem-solving. Though only a few numbers of researchers have studied this substitutive conception, studies in relational thinking have shown similar findings that it is challenging for students to relate both sides of the equal sign in solving problems (Molina & Ambrose, 2006, 2008). Therefore, the difference between recognizing the equivalence and being able to utilize such equivalence as a means for substitution should be marked.

Theoretical Rationale

Kaput (2000) proposed a movement to “algebrafying” the K-12 curriculum where he encouraged students to engage in algebraic reasoning. Carpenter and Levi (2000) answered with a proposal in re-conceptualization some mathematical topics taught in the primary grade. Following both proposals, I provide a re-conceptualization of the equal sign and equations such that students can reason with them more flexibly. The previous studies on students’ conception of the equal sign offer a solid grounding for this re-conceptualization, but they share two potential limitations.

One limitation of previous research is their lack of clarity in classifying students’ conception, or as Mirin (2019) argued, the differentiation between “operational” and “equivalent” is sometimes unclear. For instance, a primary school student who accepts the notion of 3=4-1 is considered as presenting a relational conception in Behr et al.’s study, but he/she can still have difficulties in solving equations with algebraic strategies when he/she moves into middle school. More importantly, there are inconsistencies between students’ conception of the equal sign and their problem-solving strategies. For instance, in Knuth at al.’s (2006) study, 33 students in grade-eight used the algebra method in solving the problem, but only 31 students showed relational understandings. More surprisingly, 43 grade-seven students provided a relational definition to the equal sign, but only one student used the algebra method in problem-solving. These results suggest students’ conception of the equal sign is not fully predictive of their performance in solving algebra problems. Especially in high school and above, it is likely that few students will not realize that equal sign represents an equivalence or do not believe in the legitimacy of 3=3, but their algebraic skills regarding equations can still be lacking. Therefore, to effectively extend the study of the equal sign to all k-16 education, especially for the high school and college students, it might be beneficial to reframe students’ conception of the equal sign as students’ conception of equation. In short, students’ conceptions of equation are dependent on their conception of the equal sign, but student’s conception of the equal sign is not fully predictive to their use of equation. Such expansion will bring issues, which we will address later.

The second limitation is that most studies on students’ conception of the equal sign were conducted under the context of single equation solving but lacks research on modeling students’ thinking within a system of equations. For instance, given the equation $x^2 + 2x = 1$ and ask students to evaluate $x + 1$. Besides using the standard procedure of solving for x and plugging, a student can also solve the question by re-writing the original equation as $x^2 + 2x + 1 = 2$ and taking square roots on both sides. The standard solve-and-plug method requires an equivalent understanding of the equal sign, but the later method further requires students to perform a transformation on both sides of the equation. Building off Jones's argument, I believe there is also a cognitive difference between "realizing the sameness or the substitutive nature of both sides of the equation" and "being able to transform both sides of equations flexibly in problem-solving." One could argue that those algebra skills are beyond the scope of understanding the equal sign. I contend that they are not beyond such a scope because there is an inherent kinship between students' making meaning for an equation and making meaning for the equal sign, a point we will revisit later.

The Transformative Model

In response to these potential limitations, I describe a first-order conceptual model (named the transformative model) on students' understanding of the equation as follows: In the first level, students conceive an equation as a call to execute an operation or calculation with an answer. In the second level A, students see the equation as representing the sameness of two expressions; in the second level B, students conceive equation as two parts that stay equivalent under some algebraic operation; In the second level C, students see equation as two parts that are interchangeable and used as means for substitution. In the third level, students see equation as a piece of information that only display a specific equivalent relationship explicitly but also includes a lot of implicit but inferable equivalent relationships that can be used in problem-solving. The third level is what I named the transformative equivalence conception (or transformative conception).

This model is of first-order as it is developed through analyzing my own thinking. It has two salient characteristics: Firstly, it incorporates the aforementioned research on students' conceptualization of the equal sign but extends such notion to the study of equation. Such extension is not a dramatic deviation from previous research on the equal sign, as many of them collected their data through observing students' solving equations (Alibali et al., 2007; Knuth et al., 2008). The other characteristic, which is also the central focus of this model, is the inclusion of the transformation component: conceiving equation as a piece of information that only display a specific equivalent relationship explicitly but also includes implicit but inferable equivalent relationships that can be used in problem-solving. I now use the following example to illustrate the above meanings:

Given $x^2 - 3x + 1 = 0$, find the value of $3x^3 - 8x^2 + x - 1 + \frac{3}{3x}$.

Notice to solve this problem, besides the common method of solving for x and then plugging the value, students will be benefitted from substituting "$x^2 - 3x + 1$" by 0. Students may also want to further conjecture the equations "$x^3 - 3x^2 + x = 0$" and "$x - 3 + \frac{1}{x} = 0$" from "$x^2 - 3x + 1 = 0$", and use these two new relationships to substitute x^3 and $\frac{3}{3x}$ in problem-solving. Here, I argue being able to recognize "$x^2 - 3x - 1$" as substitutivity equivalent to 0, and being able to conjecture a new equation and then recognize "$\frac{1}{x}$" $is\ substitutive\ equivalent\ to$ "$3 - x$" are cognitively different. I also postulate the second equivalence is more difficult to recognize since producing equation such as "$x - 3 + \frac{1}{x} = 0$" requires students to multiplicatively compare both expressions with respect to x, which is represented by dividing by x. Though dividing a variable to both sides is a common mathematical practice, it is often done to reduce the order or perform cancellation. However, since here dividing x will not accomplish either goal, students are less likely to use such a strategy as I illustrate in a subsequent section.

The model is also constructed with a partial hierarchical order, but it is not a linear progression where students gradually find new properties of equation. Instead, this model is an "emancipation process," where students gradually become less and less constrained to lower-level understandings. For instance, for students who solve "$x + 7 = 12$" with algebraic method in Knuth et al. (2006) study, when they subtracted 7 on both sides, they have performed a transformation, but their use of transformation can still be unnecessarily restricted and they might still be unable to perform other types of transformation in different questions. Therefore, though the model is hierarchical, the hierarchy is determined by the extent of how restricted students are using equations in problem-solving. Similarly, as Jones et al. (2012) suggested, students' development of different conceptions of equal sign does not follow a strict order, and one might develop substitutive conception before mastering an equivalent conception. Therefore, I put several conceptions as parallel, and the ordering in my model is certainly tentative rather than deterministic.

Methodology and Method

In considering my methodology, I follow radical constructivism and believe our knowledge is constrained by our experience in the sense that we do not have direct access to external realities or absolute truth (even assuming they exist). Consequently, each student constructs their own conceptualizations of mathematical ideas through their unique experiences. We do not have direct access to their understandings, but we can build hypothetical models of students' knowledge (Glaserfeld, 1995). Thompson (2013) reminded us of the importance of attending to students' meaning in mathematical activities and ensuring students understand mathematical objects productively. Conceptual analysis, as elaborated by Silverman and Thompson (2008), is then an approach in which researchers will model productive meanings of a concept such that those meanings are well connected with other mathematical ideas or students' life experiences. Educators, ideally, can then use those models to analyze students thinking and guide students to more productive understandings.

Following these beliefs, I conducted several semi-structurted clinical interviews with pre-service teachers and tried to identify possible ways that students are using or can use in conceptualizing the equal sign and equations. Research has reported the similarities between pre-service teacher and high school students in terms of their mathematics performance in non-college math topics (Moore & Carlson, 2012; Carlson, Oehrtman, & Engelke, 2010). In short, though the participants here represent a convenience sample, the results are applicable to a broader population. Here I will focus on my interview with Meki, who was a second-year undergraduate student registered in our math education pre-service teacher program. She has completed several college-level math courses but not high-level analysis courses. In the interview, I asked Meki to go through six algebra problems and explained her thinking, and most problems are algebraic questions that have multiple approaches but will be solved most efficiently by transforming the given equation and then using the substitution method. The interviews try to examine how i) interviewees' conceptualization of the equation is consistent/inconsistent with my conceptual model, especially regarding the transformative component; ii) what are the affordances and constraints of my conceptual model in modeling students' conceptualization of equation, especially regarding the transformative component.

Empirical Result

In this section, I report Meki's answer on two problems with detail. The first problem asks the student to evaluate $\frac{1}{x}+\frac{1}{y}$ when presented with the claims that $xy=1; x+y=1.$ Meki solved the problem by realizing $\frac{1}{x}+\frac{1}{y}=\frac{x+y}{xy}$. However, when I asked about her thinking, she said that she did so as she felt there was no way to directly substitute the equations (pointing to equations $xy=1; x+y=1$) into the unknown expression, and she explained that there was no "xy" or "$x+y$" appeared in the equation. I then intervened directly and told her that there was a way of doing direct substitution; she, after some thought, realized the other solution which was substituting 1 by "xy" and change the expression $\frac{1}{x}+\frac{1}{y}$ to $\frac{xy}{x}+\frac{xy}{y}$ which equals $x+y=1$. When I asked her why she didn't think of such substitution in the first place, she replied, "yeah, normally you don't put more variables into solve, cause you try to solve and get rid of the variable and find the numbers."

Meki displayed a substitutive conception of equation since she was comfortable in evaluating $\frac{x+y}{xy}$ by substitution, and also she mentioned that she started the problem by looking for some substitution. However, from the quote, I argue that her substitutive conception might be restricted in the sense that she normally does not substitute numbers by symbols. Therefore, I hypothesize that she lacks a transformative conception, and her use of equations is not flexible. Though this result does not directly substantiate the difference between the transformative conception and other conceptions, it at least suggests that a student with substitutive conception can still experience unnecessary restrictions in the potential ways of using the substitution method.

I then gave Meki the following problem, which was similar to the one I introduced earlier, and the difference is we had $\frac{3}{3x}$ in previous problem but $\frac{3}{a^2-1}$ here (which are equivalent)

Given $a^2 - 3a + 1 = 0$, find the value of $3a^3 - 8a^2 + a - 1 + \frac{3}{a^2-1}$.

In dealing with the expression $3a^3 - 8a^2 + a$, Meki started by extracting a common factor "a," and rewrote the expression into the form $a(a^2 - 3a + 1 + 2a^2 - 5a)$, which she then simplify as $a(a^2 - 5a)$. She then failed to make too much progress beyond. She mentioned she could do this trick again twice (referring to the trick of finding"$a^2 - 3a + 1$" in the expression and the substitute it by 0), but she tried some mental calculation and gave it up. In dealing with the term $\frac{3}{a^2+1}$, she rewrote it as $\frac{3}{a^2-3a+1+3a}$. She explained that since the expression "$a^2 + 1$" has two terms that were the same as the given equation's, she wanted to introduce some new terms to make it zero. It is important to mark here that she said she did not foresee the result would be $\frac{3}{3a}$, but she was trying to get rid of the two terms and had less term in the denominator.

Here I argue that Meki was using the substitutive conception, and she was working very hard to substitute the expression a^2-3a+1 by 0 in the exact form to the unknown expression, and she did not realize that one can also generate other ways of substituting (e.g. $a^3 = 3a^2 - a$) in solving the question. Though the problem can be solved by repetitively using direct substitution, the mental effort that is required in such a process was huge, and students are likely to give up. A student with a transformative conception may, however, have a very different approach to doing this problem. One approach that I observed from my cohorts was creating a "tool column" where he made a column of all equivalent forms such as $a^3 = 3a^2 - a$, and used these forms when he felt he needed it. Notice in such an approach, the student was still substituting one side of the equation with the other side, but such substitution requires a prior transformation of the original equation, and Meki seemed to be reluctant to perform it.

After I told Meki the answer and gave explicit hints on these potential equivalent equations, the students solved the problem without too many struggles. When reflecting on her thinking, she mentioned that she never did anything like that, and she said: "I was thinking a lot of it like taking things like this (circling the original equation) as it was instead of moving terms around." Her explained insistence in substituting the whole equation in its original form again suggested her use of substitution is restrictive, and a transformative conception of the equation is cognitively different from a substitutive conception.

When doing other problems, Meki presented similar thinking patterns where she is comfortable in making substitutions, but her use of the substitution method is unnecessarily restrictive. There were also some interesting findings that I noticed when I asked my cohort to experiment with some of the problems: For example, one of my cohorts realized $a^3 = 3a^2 - a$ and simplified most terms, but he struggled about $\frac{3}{3a}$. When I asked if he could conclude anything about $\frac{1}{a}$ from the given equation, he believed he could not. Certainly, by being able to simplify the higher-order terms, he displayed transformative conception, but his transformative conception does not fully support all forms of transformation (e.g., he did not notice here $\frac{1}{a} = 3 - a$).

Behr et al. (1980) summarized students' operational conception as "an extreme rigidity about written sentences, an insistence that statements be written in a particular form, and a tendency to perform actions (e.g., add) rather than to reflect, make judgments, and infer meaning." (p.16), and he named the flexible part as relational conception. Similarly, students can hold a relational conception but present the same rigidity in using algebraic methods in solving problems, which implies a lack of Jones's substitution conception. Furthermore, the student can also hold a substitution conception but present the same rigidity in using substitution, which implies a lack of the transformative conception. The case of Meki serves as a proof of such an argument where students show substitution conception but with unnecessary rigidity in the ways of substituting.

Conclusion and Compromise

Students' conception of the equal sign is important, but a differentiation between an "operational" and a "relational" is too broad to explain the wide spectrum of students' performance on operating with the equation. Consistent with Jone's finding that students can observe the equivalence between two sides of the equation while not substituting each side when helpful, I found some students can substitute each side but only perform substitutions in limited ways. In general, this study points out the variety of operations that exist in operating with equations, and students suffer from unnecessary restrictions in performing them as their conceptualization of how an equation could be operated in problem-solving is incomprehensive.

Here are two important questions to revisit: why include a transformative component of the equation into the study of the equal sign? How such an extension to the conception of equation may bring issues? For the first question, it is an observation that most studies of the equal sign are generalized through studying students' meaning of how an equation can be written/operated in problem-solving. Indeed, it is hard to imagine how one student can develop a conception of the equal sign without experiencing different ways of writing/manipulating equations. Besides, such a conception is tightly related to Jones' substitution conception, as both concern the potential ways of utilizing equations in problem-solving, and both believe there is a nonnegligible and important cognitive difference between understanding the equivalence and utilizing such equivalence in problem-solving. For the second question, since the conceptualization of equation goes beyond pure algebraic context (e.g., one may argue a complete conceptualization of equation has to contain the idea of function, as function is a special "equation" or a collection of an infinite number of "equations"), the provided conceptual analysis is incomplete. The model is created independently from concerning the students' conception of variables, the cognitive gap between arithmetic and algebra, and the rich real-life or mathematical contexts that an equation can be embedded. I concede all these factors are important points that a fully comprehensive study of equation should include. Certainly, this model is not claiming a complete analysis of students' conception of equation. Nevertheless, the primary aim of building such a model, which is to support emphasizing and studying the transformative property of equation in teaching and learning, should remain intact.

To contextualize this study into the broader field of education theory, I argue it echoes the general belief in fostering students' critical thinking and creativity as it relates to their mathematical learning. Adopting Ennis's (1996) definition of critical thinking with its emphasis on reflectiveness and making a choice from what to believe and what to do, the transformative conception may support creative reasoning and critical thinking since it invites students to think and consider all possible ways of writing an equivalent relation. Cobb(1998) raised an argument that viewing Mathematica learning as an "acculturation," and through providing more flexibility in problem thinking, the transformative conception may help students to conquer certain fixed norms in solving equations and experience the creative culture from the mathematics society (such as avoiding thinking substitution as only using numbers to replace symbols). Under such a perspective, mathematics equations and expressions become Lego blocks that students can play with and make creations accordingly, but if and only if there are multiple ways of playing, students have motivations to consider different possible mathematical operations, and what information can different equivalent expression produce. In such a way, students are potentially engaging with mathematics critically, creatively, and authentically.

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