TWO PROSPECTIVE MIDDLE SCHOOL TEACHERS REINVENT COMBINATORIAL FORMULAS: PERMUTATIONS AND ARRANGEMENTS

DOS FUTUROS MAESTROS DE ESCUELA INTERMEDIA REINVENTAN FÓRMULAS COMBINATORIAS: PERMUTACIONES Y ARREGLOS

Joseph Antonides
The Ohio State University
antonides.4@osu.edu

Michael T. Battista
The Ohio State University
battista.23@osu.edu

We report on findings from two one-on-one teaching experiments with prospective middle school teachers (PTs). The focus of each teaching experiment was on identifying and explicating the mental processes and types of intermediate, supporting reasoning that each PT used in their development of combinatorial reasoning. The teaching experiments were designed and facilitated to guide each PT toward reinventing multiple combinatorial formulas. Drawing on a subset of this data, we describe the development of the PTs' mental processes and reasoning as they came to construct formulas for counting permutations and arrangements without repetition, and we analyze our findings through a psychological constructivist framework.

Keywords: Advanced Mathematical Thinking; Cognition; Teacher Education–Preservice

Enumerative combinatorics is a mathematical discipline concerned with the activity of *counting*. More specifically, by "counting," we mean finding the cardinality of particular set, either by exhaustive listing or by using a more sophisticated technique. Researchers have taken an increased look into students' combinatorial reasoning (e.g., Batanero et al., 1997; English, 1991; Fischbein & Gazit, 1988; Lockwood, 2011; Maher et al., 2010; Tillema, 2013), but relatively little research has investigated the combinatorial reasoning of *teachers*, either current or prospective (exceptions to this include McGalliard & Wilson, 2017, and Speiser et al., 2007). Research on teachers' development of combinatorial reasoning can be an important component of studying the development of the specialized mathematics content knowledge (Hill et al., 2008) needed by teachers.

The aim of the present study was to investigate the nature of prospective middle school teachers' (PTs') combinatorial reasoning. The study involved one-on-one teaching experiments with two PTs, DC and NK. In these teaching experiments, DC and NK were guided to develop increasingly sophisticated conceptualizations and ways of reasoning needed to solve increasingly complex tasks. Ultimately, both PTs constructed multiple generalized counting formulas and ways of making sense of those formulas. We present a subset of data in which the PTs developed formulas for counting permutations and in which DC developed a formula for counting arrangements without repetition. The two tasks outlined below, as well as numerous extensions to these tasks, were instrumental in guiding the PTs' development.

- *3-Cube Towers with 3 Colors.* Using three different colors of cubes, how many different towers 3-cubes-high can be made without repeating colors? [Answer: 3!, or 6]
- *3-Cube Towers with 5 Colors.* Using five different colors of cubes, how many different towers 3-cubes-high can be made without repeating colors? [Answer: $5\times4\times3$, or 60]

The following research question guided this study: How do two PTs' conceptualizations and forms of reasoning develop as they are given increasingly complex tasks pertaining to permutations and arrangements with repetition, particularly tasks within the context of constructing block towers?

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Literature Review

Research on combinatorial thinking has consistently shown that students tend to struggle with solving counting problems. This includes, for instance, choosing an appropriate combinatorial operation for a given situation (e.g., Batanero et al., 1997; Fischbein & Gazit, 1988). Researchers have identified certain practices that help students develop in their combinatorial reasoning, such as productive listing (Lockwood & Gibson, 2016) and reflecting on the outcomes being counted (Lockwood, 2014). Tasks within the context of enumerating block towers have been shown to be particularly useful for developing combinatorial reasoning, both with children (Maher et al., 2010; Maher & Speiser, 1997) and with prospective elementary teachers (McGalliard & Wilson, 2017; Speiser et al., 2007).

One study investigated the processes by which two undergraduates (former integral calculus students) came to reinvent four combinatorial formulas (Lockwood et al., 2015), including those pertaining to this study. Framing their analysis using Lockwood's (2013) model of combinatorial thinking (sets of outcomes, counting processes, and formulas/expressions), the authors conjectured that a reinvention of counting formulas would require reasoning about counting processes—in particular, about the Multiplication Principle. Instead, the authors found their participants relied heavily on empirical patterning to reinvent certain formulas. We situate our study within the literature, guiding two PTs to construct generalized combinatorial formulas and develop in their combinatorial reasoning along the way.

Theoretical Perspectives

This study used a teaching experiment methodology (Steffe & Thompson, 2000) which was most appropriate given that the primary purpose of the study was to investigate the development of NK's and DC's conceptualizations and ways of reasoning as they were posed with increasingly complex combinatorial tasks. Ultimately, these tasks led to the construction of several combinatorial formulas and ways of using and making sense of those formulas; in this sense, our study is consistent with the principle of guided reinvention within the theory of Realistic Mathematics Education (Freudenthal, 1973; Gravemeijer, 1999). We view this work as contributing toward the overarching goal of developing a hypothetical learning trajectory (Simon, 1995) for permutations and arrangements without repetition, tracing the development of the PTs' conceptualizations and ways of reasoning as needed to make sense of increasingly complex tasks and—including the construction of generalized counting formulas.

Mathematically, we focus on the development of two combinatorial structures: permutations and arrangements without repetition. A *permutation* is a particular ordering of a set of n (distinct) objects. The number of permutations of n objects can be expressed as $n\times(n-1)\times(n-2)\times \ldots\times 2\times 1$, or $n!$. An *arrangement without repetition* is an ordering of a subset of k objects from a set of n distinct objects, the number of which can be expressed as $\frac{n!}{(n-k)!}$.

Reformulating the elaboration of Piaget's theory of abstraction by von Glasersfeld (1995) and Steffe (Steffe, 1998; Steffe & Cobb, 1988), Battista (1999, 2007) proposed *levels* of abstraction of sensory objects and motor activities (collectively called mental items). At the most basic *perceptual* level, a person has abstracted an item from their experiential flow and can perceive the item as a coherent unit. At this level, the item cannot be re-presented (visualized) without the presence of relevant sensory input. At the *internalized* level, a person can either re-present sensory objects in their mind in the absence of perceptual material or reenact a motor activity in the absence of kinesthetic signals from physical movement. However, the internalized level is limited in that a person cannot yet reflect upon an item's re-presentation or analyze an item's composition and structure. Only upon reaching the *interiorized* level of abstraction can a person reflect upon and analyze internalized items.

Interiorization "leads to the isolation of structure (form), pattern (coordination), and operations (actions)" (Steffe & Cobb, 1988, p. 337). Procedurally, a student's focus shifts from performing a sequence of actions to analyzing the meanings and results of those actions, treating the procedure as an object of reflection. Upon reaching the *second level of interiorization*, a person can perform *operations* on mental items without re-presenting or generating the material, and they can use symbols as "pointers" to the abstracted material, substituting the material with these symbols. Procedurally, second level interiorization allows a person to mentally operate on a procedure's components without actualizing the procedure using numbers. Symbols can refer to abstracted spatial components (e.g., "positions" in a generic 3-cube tower) or numerical components (e.g., a numerical procedure from reasoning about positions). At the *third level of interiorization*, a person can meaningfully represent the arithmetic/algebraic structuring of a generalized computational procedure with algebraic notation.

Findings and Analysis

In this section, we present, make inferences about, and analyze selected key events in the development of the PTs' conceptualizations and forms of reasoning.

Permutations

Episode 1. Both teaching experiments began with the 3-Cube Towers with 3 Colors task, and in both cases the PT was shown an example. NK constructed all six 3-cube towers using a single blue, green, and red cube, deconstructing the previous tower in order to make the next one.

> NK: The easiest way to think about it would be to start with a color. So, I would start with blue at the top. I always start at the top and go down. So like blue, green, red would be one, and then blue, red, green would be two.

NK then constructed the two 3-cube towers with a green top cube and the two towers with a red top cube. Notably, NK's count did not match her intuition (that there would be 3^2 towers), so she used the cubes to construct all towers in order to verify her count. Given the first follow-up task—counting 4-cube towers with 4 colors without repetition—NK responded,

> NK: OK, so same thing. Let's start with blue. So we have blue, black, green, red. I'm not gonna sit here and make all of these because, if you start with blue, you know that there's, if we have 3, then there's 6 combinations that can have blue at the top. … So blue times 6, 6 times 4, 24.

After asking for further elaboration, NK continued her explanation.

> NK: So, however many times you can arrange these three [bottom three cubes] is gonna be however many times you can arrange this whole thing, because blue will constantly be at the top. So you can just kind of omit it [the top cube] out of your thinking and see how many times this [bottom three cubes] can be combined, and then throw the blue at the top and that's the tower of four.
>
> NK then indicated the same number of towers could be made with each choice of top-cube color, motivating her multiplication of 6×4.

DC, on the 3-Cube Towers with 3 Colors task, constructed all six towers (so that all towers were present on the work-table) using a strategy similar to NK's, except his construction process was anchored by the color of the bottom cube (which he called the "base") instead of the color of the top cube. Given the 4-cube tower follow-up task (with black cubes added as the fourth color), DC deconstructed each of his original six towers; he then placed three green cubes in a line on the work table and was going to place three red cubes next to them, but he shifted his approach and instead constructed six 4-cube towers each with a green cube as base. He reasoned,

> DC: Each color has two possible towers if that color is the base [of a 3-cube tower]. But I *rose* it a level. So this is still the red base [*pointing to the two 4-cube towers with red cube second from the bottom*], but it's on top of the green base.

To clarify, DC had constructed six 4-cube towers, all with a green base and two with a red cube second-from-the-bottom. When he said he "rose it a level," he was referring to his action of taking six 3-cube towers and adding a green cube to each. After this, DC constructed all six 4-cube towers with a red base. He then predicted there would be the same number of towers with blue and black bases, concluding there would be $6 \times 4 = 24$ total 4-cube towers. Later, DC further explained, similar to NK, that 4-cube towers with green bases can be made by taking the composite of six 3-cube towers and appending a green cube to the bottom of each tower.

Inferences. We infer that NK conceptualized towers spatially as composites consisting of a single cube appended to an (n-1)-subtower, while DC had a similar spatial structuring but with the appending cube in the bottom position rather than the top. This spatial structuring led the PTs to organize their processes of tower construction by using the appended cube as an anchor, similar to what English (1993) calls a "major constant item." When transitioning to the 4-cube-tower follow-up task, both PTs used a recursive strategy, now reconceptualizing each 3-cube tower as a composite unit and operating on the composite of six 3-cube towers (mentally, in NK's case, or perceptually, in DC's case) by appending to each tower an additional cube. We also infer that, initially, DC planned to construct each 4-cube tower systematically, but he realized the composite of 4-cube towers could be constructed by building on the composite of 3-cube towers.

Analysis. Both NK and DC had interiorized the process of constructing 3-cube towers, indicated by the fact that their constructions were coordinated by a spatial structuring. Further, we interpret NK's reasoning on the 4-cube-towers task as reasoning about "symbolic" positions (of a generic 4-cube tower) rather than about specific instances of towers, indicating second level interiorization of the process of constructing 4-cube towers. DC's reasoning was also coordinated by a spatial structuring, but his reasoning relied on the perceptual material available on the worktable and did not appear to draw on the structure of a generic 4-cube tower. Thus, we interpret that DC interiorized the process of constructing 4-cube towers.

Episode 2. *NK.* Extending to counting 5-cube towers with 5 colors of cubes, and without constructing any towers perceptually, NK reasoned,

> NK: So you could do the same thing where you start with a different color, but how many different combinations you can make with 4 is going to be 24. …Let's pretend [the fifth color] is white. So you could start with white, and you would have 24 different combinations of these [gestured to other four colors of cubes] that would go under the white. So it would be 24 times 5, because each one of these colors would get their chance to be at the top.

Extending to counting 6-cube towers with 6 colors, NK described her strategy more generally.

> NK: So there's a pattern. So it's, whatever 5 is, which we found that 5 is 120, it would be 120 times 6. … Sorry, I keep thinking with the mindset of, just like, you can omit the top cube. So you can start with like a constant, your constant would be like whatever color [is at the top].

The next follow-up task jumped to counting 9-cube towers with 9 colors. She did not have an immediate answer, but she took colored pencils and paper and started reflecting on her previous enumerations, searching for a pattern.

> NK: Oh! Okay, okay. So, if you were to start with two colors, there are two combinations. Then you go to three, we found that there are six. Then you go to four, and find whatever 6 times 4 is, which is 24. Then you take 24 and multiply it by 5. 'Cause we have 5 things, so I multiplied it by 5 and got 120. And we came here, and we multiplied 120 by 6, which is… 720. And then you can

> take that and multiply it by 7… which is 5,040. And then you can do that and multiply it by 8, which is the number of combinations you get if you had 8 colors, which is 40,320. And then you can multiply it by 9 to finally find how many it would be, which would be 362,880.

Inferences. As with her spatial structuring of 4-cube towers from 3-cube towers, NK visualized appending a fifth-color cube to the top of each 4-cube subtower, then reasoning multiplicatively with the number of available colors. Similar to her reasoning in Episode 1, NK counted 5-cube towers by reasoning about positions—that is, her reasoning drew on an abstraction of the spatial structure of a generic 5-cube tower. With this spatial structuring guiding her reasoning, NK used a recursive strategy once more, multiplying the number of top-cube color possibilities by the number of ways to arrange the cubes in the (n-1)-cube subtower.

Analysis. NK used an interiorized spatial structuring to guide her reasoning about the number of towers of any height, although at this point her reasoning was recursive in the sense that she relied on knowing the number of (n-1)-cube-towers to compute the number of n-cube-towers. As with the previous episode, NK's reasoning is based on positions, indicating second level interiorization of her recursive scheme for counting n-cube towers.

Episode 2. *DC.* Counting 5-cube towers with 5 colors, DC searched for a pattern in the number of towers that are present with a given color base cube.

> DC: I'm going to take a guess. So, we went from two of each to six of each, which means we multiply by 3. So when we multiply by 3 again… oh no, I should multiply by 4. I'm gonna multiply that [pointed to the six constructed 4-cube towers, each with a green base] by 4 because there are four colors, excluding the new color. I don't know if that's right…. So, 6 times 4, 24 combinations. No, 24 is what we have now if we did just have the 4 of them.

He placed the six 4-cube towers with green bases in one group, and adjacent to those he placed the six 4-cube towers with red bases. He then lined up six blue cubes and six black cubes to symbolize the 4-cube towers with blue and black bases, respectively. With this, he could see there would be 24 4-cube towers in total. Although he still expressed uncertainty, DC predicted there would be 24×5 5-cube towers that could be made with 5 colors.

To help DC resolve his uncertainty, he was given five different colors of cubes arranged in a line on the table and was asked how many different ways the five cubes could be rearranged by interchanging cubes. DC exhaustively enumerated 24 of the possible 5-cube orderings, keeping the right-most cube (which he still called the "base") constant and multiplying, at the end, by 5. His enumeration was systematic although not a complete odometer pattern (English, 1991). Moving to counting 6-cube orderings in a row, DC reasoned,

> DC: I know that in the 5-color scheme, there are 120 combinations per base color. Now, with a *super-base* sixth color, there are six different options for the super-base, and 120 options for the regular base. And so 6 times 120 is… 720.

Of note, DC's verbal comments indicated he thought about constructing towers and enumerating orderings of cubes interchangeably. Extending to counting 9-color orderings in a row, DC said,

> DC: I would go through if we had seven colors and then eight colors first. All I would do is take that 720… multiply it by 7, to get 5,040. That's how many combinations we have for seven colors. Then multiply that by 8… 40,320. And then for nine, we would multiply that value by 9.

Inferences. DC had constructed 12 of the 24 4-cube towers with 4 colors, and he placed six blue and six black cubes on the table to represent the remaining 12 towers. With this visual aid, DC predicted there would be 24×5 5-cube towers, but he did not seem to visualize "raising" this 24 5-cube-tower composite with the added sixth color of cubes (as he did to construct 4-cube towers from

3-cube towers). After moving to counting 5-cube orderings, DC's counting was systematic, and he reflected on this process to enumerate orderings of larger numbers of cubes.

Analysis. At first, DC seemed to have internalized, but not interiorized, the process of constructing 5-cube towers. After perceptually finding all 5-cube orderings with a fixed right-most cube (and reflecting on those actions), DC reasoned about positions—specifically, multiplying the number of 5-cube orderings with a fixed right-most cube (24) by the number of possible right-most cubes (5)—which indicated second-level interiorization of the process of constructing 5-cube towers. With subsequent tasks, his reasoning extended to counting increasingly larger orderings, indicating second level interiorization for a more general (recursive) scheme for counting permutations, although it became less apparent that DC's reasoning was guided by his spatial structuring.

Episode 3. Extending to counting 20-cube towers with 20 colors, NK began to search for a function that would produce the number of towers of a given height (and number of colors), n. Thinking the function would be exponential, but unable to predict a specific formula, she reflected on her previous strategies.

> NK: So you would start at 20, so it would be 20 times 19 times 18 times 17 times 16 times 15 times 14 times 13, 12, 11, 10, 9, 8, 7, 6, 5, 4, 3, 2, 1. The whole time I've been multiplying *this* direction [motioned from left to right, indicating starting at 1 and ending with 20], but I just started multiplying by the number of colors we have.

On follow-up tasks, NK then predicted the number of 100-cube towers with 100 colors to be $100\times 99\times 98\times \ldots\times 1$, and she wrote a general expression for the number of n-cube towers with n colors to be $n\times(n-1)\times(n-2)\times(n-3)\times \ldots\times 1$.

DC predicted the number of 20-cube towers with 20 colors by multiplying the number of 9-cube towers, 362,880, by 10, then by 11, by 12, etc., and finally by 20. He searched for an "equation" that would calculate the number of towers, but he was not able to find one. Extending to counting 100-cube towers, DC described that he would continue multiplying the product found in the 20-cube-tower task by 21, by 22, by 23, etc., until finally multiplying by 100. After leading DC through a quick review of his work so far, DC said,

> DC: *Oh wait*! Would this work, where we have 2 there, and that's the 2 combinations we can get with 2 colors…. Then we multiply that by 3, because then we added a third color. And then we would multiply that by 4, because we added a fourth color. Then we multiplied that by 6, which is what I'm doing, but then you could just go up to 100.

He wrote the following to express what he had verbalized: $(((((2)3)4)5)6\ldots)100$. He initially kept the parentheses so as to maintain the order of multiplication, but he later cited the associative property of multiplication to reason that he could remove the parentheses. Finally, extending to counting n-cube towers with n colors, DC said there would be $2\times 3\times 4\times \ldots\times n$ different towers.

Inferences. NK's calculations, now reversed from their original order, no longer seem to be guided by a mental process of recursively operating on towers of height n-1 to form towers of height n. For DC, after guiding him through a review of his previous enumerations, he reconceptualized 362,880 as the product of consecutive integers ($2\times 3\times 4\times \ldots\times 9$). He then extended and generalized this strategy, although at this point his reasoning still seemed recursive.

Analysis. Both NK and DC reached the third level of interiorization for counting permutations, indicated by both PTs' generalized algebraic expressions. Even still, NK's reasoning seemed more sophisticated than DC's as she realized she could reverse her multiplicative process and achieve the same result, which we suspect is important for counting and reasoning about arrangements.

Overall, both PTs exhibited remarkable progress in their conceptualizations and ways of reasoning about permutations. NK and DC were able to enumerate block tower composites successfully and efficiently with cubes, either mentally or with perceptual material, and use this to structure their numerical reasoning.

Arrangements—DC

Due to space constraints, only a subset of DC's progress toward thinking about arrangements without repetition is presented.

Episode 4. DC first considered the task of counting 3-cube towers with 4 possible colors of cubes. He systematically constructed the six 3-cube towers with a red base, then reasoned:

> DC: If we use red as our first bottom color, our first base, after red only 3 other colors can be the middle color in our tower of 3. Then, for each of those, there are only 2 colors remaining. And since it's a tower of 3, it's just those 2 colors on top. … So 6 total for red, times 4.

He clarified "times 4" was to account for the number of possible base-cube colors. Given the 3-Cube Towers with 5 Colors task, he constructed the remaining 3-cube towers with a red base, including those with the newly-added fifth color, culminating in 12 3-cube towers with a red base on the worktable.

> DC: So now there are 12 different combinations for each color as base, times 5—60. There are 60 possible combinations.

Given the next follow-up task, counting 3-cube towers with 6 colors, DC initially tried to find a pattern, first noticing a doubling pattern in which the number of towers "per base" went from 6 (using 4 different colors) to 12 (using 5 different colors). DC predicted there would be 24 towers per base in the 6-color task, but he was not certain. He resorted to systematically constructing all 3-cube towers with a red base using 6 colors, finding 20 towers in total. He reflected on the number of towers that were added when transitioning from towers with 5 colors to towers with 6 colors, but this did not lead to a new insight. We returned to the task of counting 3-cube towers with 4 colors, and it was in this task that DC made a new insight.

> Int: With the first problem with 4 colors making towers 3-cubes-high, I remember you saying in this original problem that you thought about it as 3 times 2…. So, you saw the 3 times 2 in that problem. Could we see something similar when we move up to 5 colors?
>
> DC: I mean, it's 4 times 3. 'Cause there are 4 possible options for the second tier, and then on top of each of those there are 3 possibilities for what can be left…. That's 12 per base, and then 12 times the possible 5 bases is 60 combinations. With 6 colors, if we look at it in the same way we just did, there are 5 combinations for what can come on top of each base, times 4 possibilities for what can come on top of that second tier… 5 times 4 is 20, times the possible 6 bases, which is 120 possible combinations. So, if we were dealing with 4 colors—*Oh!* Here it is, I think I might have just done it. We have n times $n - 1$ times $n - 2$ equals that value…. This gives us the number of possible combinations above each base, times the number of bases, gives us the total number.

Inferences. DC's reasoning was guided by a specific spatial structuring: He conceptualized composites of 3-cube towers by focusing on a subset with a particular base-cube color appended to 2-cube towers, then generated the entire composite of 3-cube towers by multiplying by the number of possible colors for the base. When the teacher-researcher led DC to revisit and reflect on his numerical procedures, he realized a numerical pattern connected to his spatial structuring: the number of color possibilities "on top of each base" multiplied by the number of color possibilities "on top of that second tier."

Analysis. DC's process of counting 3-cube towers reached second-level interiorization, indicated by his reasoning about positions within 3-cube towers, and quickly progressed to third-level interiorization as he generalized his numerical procedure to counting 3-cube towers with any number of colors.

Episode 5. DC was asked to count k-cube towers with n different colors, without repetition.

DC: Let's just say *k*=5 in this instance. Then the equation would be *n*(*n*-1)(*n*-2)(*n*-3)(*n*-4). And… the integer that you're subtracting is *k* minus—like, goes up to *k*-1, but doesn't quite get to k. And so, it's like the reverse factorial here. Like, this is *k*-1 [pointed to 4 in *n*-4], *k*-2 [pointed to 3 in *n*-3], *k*-3 [pointed to 2 in *n*-2], *k*-4 [pointed to 1 in *n*-1].

When asked to generalize from k=5 to a general k, DC used this same "reverse factorial" idea and wrote $n(n-k-(k-1))(n-k-(k-2))(n-k-(k-3))\ldots$. He was asked if $(n-k-(k-1))$ might be able to be simplified; after some algebraic manipulation, he realized that, for any value of k, the number of towers of height k with n colors of cubes is $n(n-1)(n-2)(n-3)\ldots(n-(k-1))$.

Inferences. DC thought that a generalized expression for counting k-cube towers with n colors would require an expression involving both n and k, leading to his conceptualization of the "reverse factorial."

Analysis. DC's process for counting arrangements had reached third-level interiorization.

Conclusions

The PTs' spatial structuring was instrumental toward guiding the development of their reasoning about permutations and arrangements. Both PTs developed recursive strategies but progressed past recursion through processes of action, reflection, and abstraction. This ultimately led their processes of enumerating permutations and arrangements to third-level interiorization.

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