ESTABLISHING A CARTESIAN COORDINATION IN THE ANT FARM TASK: THE CASE OF GINNY

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In this report, we present how one prospective elementary teacher (PT) engaged in the Ant Farm Task, which we designed to investigate PTs' reasoning about coordinate systems. We highlight the cognitive resources the PT drew upon in solving the task via the establishment of a Cartesian coordination and consider educational implications.

Keywords: Geometry and Geometrical and Spatial Thinking, Representations and Visualization, Modeling, Coordinate System

From fifth grade onward, students and their teachers are often expected to represent and reason about various mathematical concepts (e.g., geometrical shapes, functions, etc.) using the Cartesian plane. However, coordinate systems are often taken for granted; meaning, students are assumed to develop proficiency in using this representational tool in relatively unproblematic ways, and teachers are assumed to have developed understandings of coordinate systems capable of supporting their students' mathematical activity. Additionally, textbooks and curricular standards (e.g., CCSSM) describe how to draw and use a Cartesian plane but rarely provide motivation for establishing a Cartesian coordination. Generally, the rules of "generating" a Cartesian plane are given with minimal explanation for *why* we construct it in such a way or *why* using an ordered pair of numbers locates a point. In this report, we present how one prospective elementary teacher (PT) engaged in the Ant Farm Task (AFT), a task we found helpful in motivating a Cartesian coordination. We highlight the cognitive resources the PT drew upon in solving the task via establishing a Cartesian coordination and consider educational implications of the task and our findings.

Theoretical Framing

By *coordinate system* we mean a representational space in which an individual systematically coordinates quantities (Thompson, 2011) to organize some phenomenon. A coordinate system does not represent by itself; it must be created and interpreted by a cognizing subject (cf., von Glasersfeld, 1987). Put differently, we consider coordinate systems to be constructed by an individual in goal-directed activity. Relatedly, we have distinguished between two types of coordinate systems depending on the goal they serve: spatial and quantitative (Lee, Hardison, & Paoletti, 2020).

Spatial coordinate systems are used to quantitatively organize a space in which a phenomenon is situated. Constructing a spatial coordinate system involves (mentally) overlaying a coordinate system onto some physical or imagined space being represented and tagging objects within that space with coordinates. For example, consider a Cartesian plane overlaid onto an amusement park from a bird's eye view where the axes are aligned with two streets in the park. On the other hand, quantitative coordinate systems are used to coordinate sets of quantities by constructing a geometrical representation of the product of measure spaces. Constructing a quantitative coordinate system involves an individual extracting quantities from the space in which a phenomenon occurs and projecting them onto a new space, different from the space in which the quantities were originally

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conceived. For example, imagine a Cartesian plane with one axis representing the amount of wait time for a ride in the amusement park and the other axis representing the number of people in line for that ride.

Both coordinate system types involve coordinating quantities. Our use of *spatial* and *quantitative* as modifiers is intended to foreground the different mental actions involved in establishing a coordination of quantities in each type of coordinate system. We use the Cartesian plane as an example to illustrate this distinction. When constructing a *spatial* Cartesian plane, an individual starts with a space they want to organize. In order to quantitatively describe the location of objects or points within this space, the individual can establish a reference point and orthogonal lines through the reference point and use these frames of reference to describe each point's location in terms of its horizontal and vertical distance from the reference point. In other words, the individual can establish a Cartesian coordination via decomposing the location of a point along two orthogonal lines in relation to a reference point (see Figure 1a). In this case, the point's location is conceived of as a logical multiplication (Piaget et al., 1960) of the horizontal and vertical displacement from the reference point.

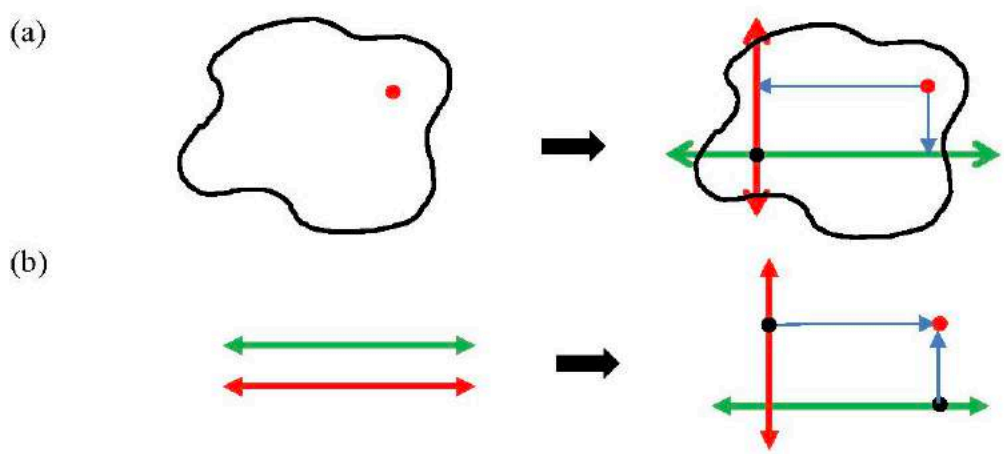

Figure 1: Model of the operations involved in establishing (a) a spatial Cartesian coordination and (b) a quantitative Cartesian coordination

In contrast, when constructing a *quantitative* Cartesian plane, an individual starts with two quantities they have disembedded (Steffe & Olive, 2010) from the space in which a phenomenon occurs (i.e., the individual has extracted them from the situation while maintaining an awareness of the quantities within the situation). Overlaying the quantities onto two number lines, and arranging the number lines orthogonally, the individual can produce a two-dimensional space, different from where the quantities were originally conceived. Finally, a point is constructed as the intersection of the perpendicular projections from each point/quantity on each number line (see Figure 1b) and a quantitative Cartesian coordination is established. In this case, the point is conceived of as a multiplicative object holding both quantities' values simultaneously (Saldanha & Thompson, 1998).

Students' Cartesian Coordinations in Literature

Several studies provide insights into students' Cartesian-like coordinations described above in graphing contexts. For example, Nemirovsky and Tierney (2004) presented a situation to Rose, an eight-year-old, in which blocks were added to or taken away from a paper bag over time. Rose was asked to show how the number of blocks in the paper bag changed over time using a line marked with Start and End. Rose produced a curve (Figure 2a) to represent the change in quantity over time.

This example shows a child spontaneously using the space above the line to show change in quantities over time. DiSessa et al. (1991) and Sherin (2000) presented a scenario describing the motion of a motorist over time to middle and high school students and asked them to produce a picture describing the motorist's motion. In solving this task, students tended to use a horizontal line representing the road and made marks such as dots and line segments to represent quantities (e.g., speed). Gradually, the horizontal line transformed into a number line representing time and some students produced a graph such as that in Figure 2b to describe the motorist's speed over time. This example demonstrates how middle and high school students utilized the vertical distance from a horizontal line to represent change in speed over time. Collectively, these examples illustrate how students, starting with a horizontal line, can use a vertical dimension to represent change in quantities over time and hence construct Cartesian-like systems. However, they do not explain how students might establish a Cartesian coordination starting with two (number) lines.

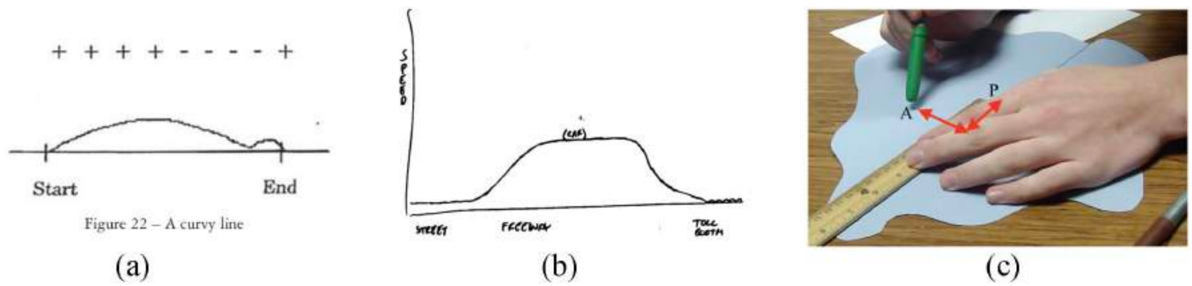

Figure 2: Two examples: (a) Rose's "time line" representation in Nemirovsky & Tierney (2004), p. 42 (b) Carl's representation in Sherin (2000), p. 432

In her previous work with four ninth-graders, Lee (2017) examined students' constructions of spatial coordinate systems and observed students establishing a spatial Cartesian coordination. For example, when asked to locate a missing person (point *A*) in reference to a rescuer (*P*) on a map, one ninth-grader described the location of *A* in relation to *P* by decomposing the straight motion from *P* to *A* into a horizontal and vertical movement from P to A (see Figure 2c). After observing students' constructions of spatial coordinate systems, we were motivated to explore how students might leverage their ways of coordinating spatially in order to coordinate quantitatively via the AFT (Lee & Hardison, 2016; 2017). The findings we present in this report extend the literature base by (a) examining PTs' constructions of coordinate systems which can inform educational support for PTs, which are scarcely documented and (b) identifying the cognitive resources that may be leveraged to engender a spontaneous Cartesian coordination starting with two (number) lines.

The Ant Farm Task

In previous work, we have hypothesized that spatial coordinations necessarily precede quantitative coordinations (Lee & Hardison, 2017). We designed the AFT as a possible way of bridging these two types of coordinations. In contrast to the above tasks, we designed the AFT to (a) be entirely situated in a spatial context in which PTs could leverage their spatial coordination, (b) start with two given lines, and (c) have the potential for engendering some of the mental actions involved in establishing a quantitative coordination. In the AFT, PTs were provided with two transparent tubes representing two ant farms (Figure 3a) and asked to imagine that each contained exactly one giant ant moving around haphazardly. Additionally, we provided a model of this situation in a dynamic geometry environment (DGE; Figure 3b); the DGE sketch contained two long, thin rectangles (ant farms), each containing a point (ant) moving haphazardly. The points' movement could be paused/activated by action buttons, and the rectangles could be moved or rotated within the DGE. Given this scenario and the DGE sketch, we asked PTs, "Can you make a single point to show the locations of both ants at any moment in time?" Following this prompt was an explanation that if we were to hide both ants, they should be able to use their new point to determine the location of the two hidden ants. We presented the spatial situation without referencing any quantities explicitly (e.g., an ant's distance from the end of the tube). One possible solution to the AFT involves establishing a conventional Cartesian coordination. Through the AFT, we investigated (a) how PTs might modify their ways of coordinating spatially to coordinate quantitatively and (b) what cognitive resources PTs draw upon when constructing a two-dimensional coordinate system from two one-dimensional lines.

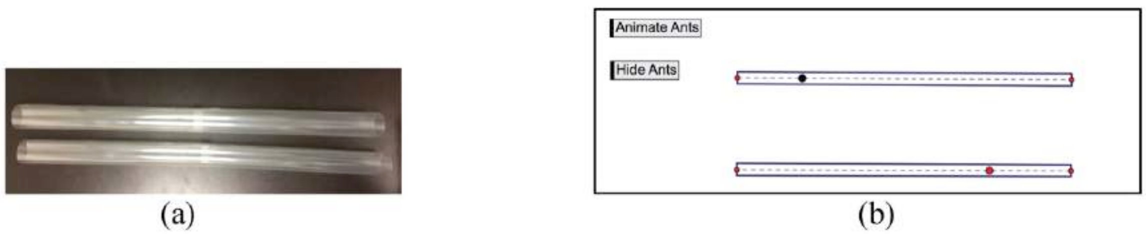

Figure 3: Ant Farm Task (a) plastic tubes representing the ant farms and (b) accompanying dynamic geometry environment sketch.

Methods

We draw on data from a teaching experiment (Steffe & Thompson, 2000) with four PTs. The overarching goal of the teaching experiment was to investigate how PTs construct and reason about coordinate systems, as well as how PTs' ways of thinking changed throughout the teaching experiment. All four PTs were enrolled in an elementary or middle grades teacher preparation program at a university in the southern U.S. PTs participated in eight 60-minute long teaching sessions, which were conducted individually or in pairs. In this report, we present and analyze data from one teaching session wherein one PT, Ginny, and her partner, Hermione, solved the AFT. We focus our analyses on Ginny with occasional remarks regarding Hermione as appropriate. We focus specifically on Ginny because her solution to the AFT contained features common to other PTs' activities on the task and a unique feature—introducing number lines.

Data Sources and Analysis Methods

Investigating PTs' mathematical thinking, which is not directly accessible, requires making inferences from PTs' observable activities. Therefore, the models of Ginny's thinking we build are second-order models (Steffe & Thompson, 2000) of what we infer from her visual illustrations, verbal descriptions, and physical gestures. For each teaching episode, we collected video recordings of PTs' actions, a screen recording of PTs' activities in the DGE, and digitized written work.

We conducted both on-going and retrospective analyses and modeled PTs' constructive activities (Steffe & Thompson, 2000). On-going analyses involved testing and formulating hypotheses during the teaching experiment based on ways PTs engaged in each teaching episode. We inferred, from PTs' engagement, instances that corroborated or contraindicated our hypotheses. After the completion of the teaching experiment, we re-visited the data corpus to do an in-depth retrospective analysis. The retrospective analysis involved four main activities that collectively refined the initial explanatory models developed during the teaching experiment. The four activities were (a) watching the entire video set or subsets of the video holistically without interruption to observe recurring patterns in PTs' activities or shifts in their reasoning, (b) identifying instances that offered insights in building working models of the recurring patterns or shifts in their reasoning, (c) constructing annotated transcripts of such instances with rich descriptions of PTs' actions, and (d) constructing/refining explanatory models of PTs' constructions of coordinate systems. Specific to the AFT, we analyzed the cognitive resources PTs drew upon when constructing a two-dimensional system from two one-dimensional lines.

Findings

We present our findings regarding Ginny's reasoning about the AFT in four phases. We highlight Ginny's activities in each phase and analyze the cognitive resources that supported her in establishing a Cartesian coordination.

Attending to Variability and Locational Simultaneity by Superimposition

When first posed with the prompt in the AFT, Ginny reiterated the problem, asking, "So, we need to make one point *while they're moving* to show where they are while *they're moving*? *At any point in time*?" to which her partner Hermione added, "I think it's hard because they're moving around a lot." We took these initial comments to indicate that both PTs were attending to, and perturbed by, two things: variability in the ants' locations and locational simultaneity required in the desired single point.

Ginny and Hermione proposed some ideas to address these elements. For instance, Hermione suggested using the mid-points of each tube, since "the ants always pass the middle." However, both PTs noticed that the ants were not always at the middle of each tube. Instead, Ginny superimposed the tubes in various ways to observe where the ants crossed each other. Specifically, Ginny aligned the two ant farm rectangles on top of each other in the DGE so that one rectangle was perfectly overlaid onto another and animated the ants. Next, she observed instances where the ant points occupied the same location at the same time. She identified three such points and claimed that one of those could be used as the desired point. However, Ginny acknowledged that these three points do not capture all of the possible ants' locations.

Seeking alternative solutions, Ginny joined the ends of the two plastic tubes, making the two tubes into one long tube, and explained, "You know both of the ants are on the same path. In the direct center, I guess that would be the one point to describe where they both are," as she drew a circle and point in the circle's center. As such, Ginny assumed the ants were moving in the center of each tube and, taking a cross-sectional view, identified one point as a projection of both ants in each tube.

From her activities, we infer that superimposing and attending to the intersection of the two points was a way for Ginny to account for locational simultaneity. When she superimposed one tube onto another, she accounted for locational simultaneity for three different instances in time; when she superimposed the tubes by joining their ends, she accounted for simultaneity for all instances in time, but from a different perspective. Although these points did not indicate where the ants were in each tube, Ginny demonstrated flexibility in taking different perspectives in viewing the ants and tubes. Collectively, Ginny's activities indicated that she viewed the tubes as objects she could manipulate and rearrange to serve her goal; they were not fixed objects, which we viewed as a critical cognitive resource in her thinking.

Establishing the Single Point as Being Dynamic

Approximately 20 minutes into the session, Ginny placed the tubes in the DGE perpendicularly (see Figure 4a) and explained she wanted to see if the ants met in the middle, where the tubes intersected. With the ants animated in the DGE, Ginny and Hermione observed that the ants crossed each other at the middle, but Ginny was unsure how to proceed: "I still don't think [we can come up with just one point] because they're never, they're not always in the same place at the same time." Wondering if the PTs conceived of the desired point as being static, the teacher-researcher (TR) asked, "So, what if somehow that point might move appropriately with the ants?" To which Hermione expressed, "So, you're saying the point can move now?" Hermione's response suggested she had interpreted the initial prompt as requesting a static desired point and she was now considering whether the desired point could be dynamic. Although this interaction was occasioned by the TR, we view this interaction as critical because the PTs established the desired point as potentially being dynamic.

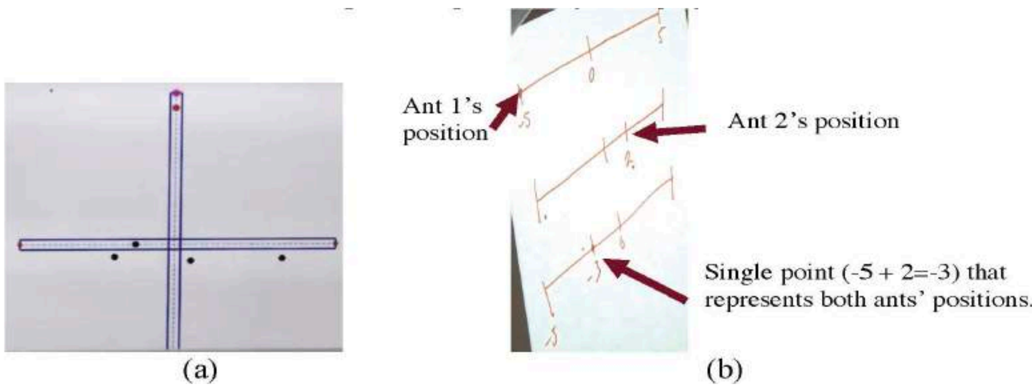

Figure 4: (a) Tubes arranged perpendicularly in the DGE and (b) Ginny's three number lines.

Using Number Lines

Approximately 30 minutes into the session, Ginny proposed a new idea: "What if we made the edges of the tubes a number line and the middle of the tube zero. So this [*pointing to the left side of a tube*] would be the negative side and this [*pointing to the right side of the same tube*] would be the positive side and this [*pointing to the center of the tube*] would be zero." Next, Ginny drew three number lines as shown in Figure 4b, explaining that the first and second number line each represented the ant's location in Tube 1 and Tube 2, respectively. Picking two locations as examples, she explained that if Ant 1 is at –5 and Ant 2 is at 2, then the single point –3 on the third number line, obtained by adding –5 and 2, should represent both ants' positions. When asked what the third number line was, Ginny explained, "[It is] corresponding to where the point is on, it would be I guess both of the tubes." As such, instead of superimposing one ant farm onto the other, Ginny created a third object to account for locational simultaneity.

Up to this point, we hypothesized that Ginny overlaid a number line onto each tube, and thus constructed a one-dimensional spatial coordinate system. Relatedly, there were two hypotheses to be tested: (a) whether Ginny's number lines were viewed as number lines superimposed onto the spatial situation, and (b) whether the numbers on her number lines were conceived of as distances (e.g., from the edge of the tube) or numerical values labeling each ants' location.

In Ginny's three number line representation (Figure 4(b)), although simultaneity was accounted for, the variability of the ants' positions yielded a non-unique point on the third number line. Relatedly, the TR asked, "what if Ant 1 is at –4 and Ant 2 is at 1?" Almost immediately, Ginny drew two perpendicular lines and explained, "So, if we had a graph and we do... negative five and then two, this point right here [*plotting the point in Figure 5a*] would describe where they are. So, instead of doing the adding them together you would graph it on a graph." She further explained that the point she plotted showed that the black ant is at 2 and the red ant is at –5. At this point, the two tubes in the DGE placed in front of the PTs were positioned perpendicularly (like in Figure 4a). We account for the sudden shift in her thinking to (a) Ginny's attention to locational simultaneity and attempting to capture both ants' locations and (b) recalling her previous graphing experience from the image on the DGE.

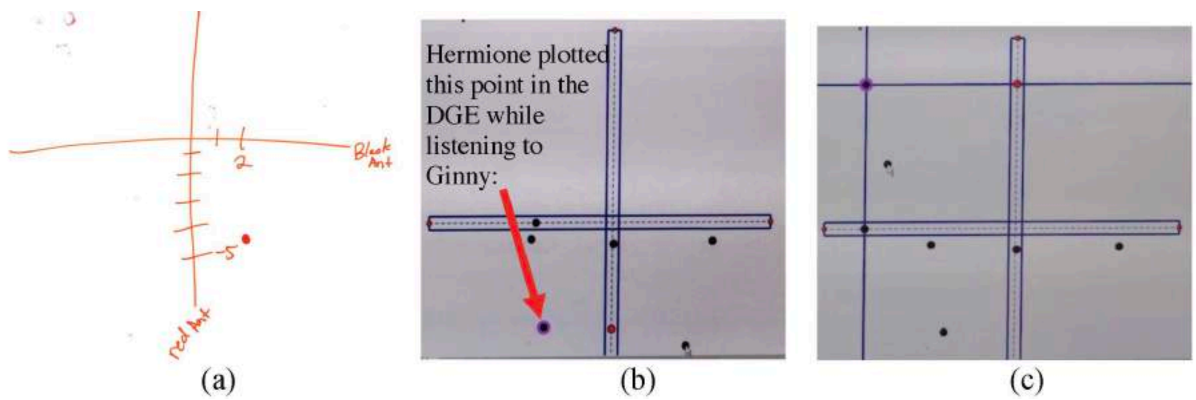

Figure 5: (a) Ginny's drawing of number lines arranged perpendicularly; (b) Hermione's point in the DGE; (c) Ginny's construction of the single point.

While Ginny was showing her new solution for finding the desired point, Hermione plotted the point shown in Figure 5b in the DGE and asked Ginny if the point she plotted would be the single point. Ginny explained "So, this [*pointing to her drawing in Figure 5a*] would be a totally different graph... so you'd get information from where the ants are on the number line on the tube. Then you would graph it on a different paper. So, it's not like a point on these [*pointing to the plastic tubes*] it's like a point on this [*pointing to her paper*].

Ginny's explanation in response to Hermione's question indicated Ginny has constructed number lines, disembedded from the Ant Farm situation, and arranged them to produce a different space from the original Ant Farm space. Thus, we inferred she constructed a quantitative coordinate system with her number lines in the sense that she viewed her "graph" as a space different from the space in which the number lines were initially conceived. Also, by the way she explained "getting information from where the ants are on the number line on the tube," we inferred her number lines consisted of numerical values indicating location in reference to the middle of the tube and that her number lines were not yet spatial coordinate systems superimposed onto the spatial situation.

Establishing the Cartesian Coordination

Noticing Ginny's differentiation between her "graph" and the ant farm space, approximately 35 minutes into the session, the TR commented, "I'm curious, so it sounds like you see this as a different space from that space, is what it sounds like." Ginny responded, "Yeah, I mean, I guess you can make it the same space but..." and sat in thought looking back and forth at the DGE screen and her sketch in Figure 5a for approximately 7 seconds. She then continued, "I guess you *could* make it the same space because...if put a number line on this," pointing to the ant farm tubes in the DGE. Pointing to each ant farm tube in the DGE, Ginny further explained, "So this [*referring to the horizontally placed tube*] would be your x and your y [*pointing to the vertically arranged tube*] of your graph and your dots [*referring to the ants*] would be your values and you just need to connect them to make your point [*moving her fingers in the air consistent with the blue arrows in Figure 1b*]."

Finally, Ginny constructed a line through Ant 1, perpendicular to Tube 1, and a line through Ant 2, perpendicular to Tube 2 and indicated that the intersection of the two lines would show the location of both ants in each tube (Figure 5c). After the TR hid the two perpendicular lines, both PTs verified their new method worked by considering several positions of ants by hiding the ants, animating the single point, guessing where the ants should be within each tube, and then checking the ants' locations. When the TR asked why they think this method works in general, Ginny explained that by placing the tubes perpendicularly, a horizontal line can be used to describe the red ant moving vertically and a vertical line to describe the black ant moving horizontally. In conclusion, Ginny established a Cartesian coordination leveraging her spatial coordinations.

Conclusion and Discussion

From Ginny's activities in the AFT, we highlight three cognitive resources we see as critical for establishing a Cartesian coordination given two lines that would enable holding a sustained image of two locations simultaneously for arbitrary positions of points on each line. First, Ginny's *attention to variability in the ants' locations* coupled with *imagining the single point as moving along with the two ants* was a critical development during the teaching session. Second, Ginny *recognized the tubes (or number lines) as objects that could be manipulated and rearranged* which supported her to arrange them in a particular way (e.g., perpendicularly) so that the locations of each ant could be accounted for simultaneously. Third, *drawing from her spatial coordinations*, Ginny utilized the two-dimensional space outside of the one-dimensional tubes spaces to construct a point outside of the tubes (or number lines). Specifically, she projected vertically from the horizontal tube in which the ant moved horizontally and projected horizontally from the vertical tube in which the ant moved vertically. Supported by these cognitive activities, Ginny successfully constructed a single point that simultaneously captured the location of both ants.

For Ginny and Hermione, devising a system to coordinate the location of two points using a single point appeared novel, meaning that establishing a Cartesian coordination by rearranging lines orthogonally and projecting from the two lines was non-trivial despite their prior school experiences. Recall that although Ginny eventually constructed a system she called a "graph," she initially viewed it as different from the ant farm space. We suspect this was due to her past experiences with coordinate systems focused predominantly on quantitative coordinate systems. Thus, viewing the Ant Farm space as a space analogous to her coordinated number line space was an additional critical realization for Ginny in solving the AFT.

Given the preceding findings, we close with two considerations: one limitation and one direction for further study. First, the wording of the initial AFT prompt may have hindered PTs from viewing the single point as being dynamic; therefore, variations of the prompt might be considered in future implementations. Second, the cognitive resources presented here that foster the establishment of a Cartesian coordination may be specific to Ginny and the AFT, which require further research with more PTs. Future research can also look into what engendered Ginny's transition from one phase to another (e.g., TR moves) and how these can be leveraged to support PTs and consequently their future students' constructions of coordinate systems.

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