MENTAL MATHEMATICS IN THE CLASSROOM: CONTENT, PRACTICES AND PAPERT'S *MATHLAND*

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This lecture reports on aspects of a larger research programme focused on studying mental mathematics in elementary and secondary mathematics classrooms. It specifically addresses an unplanned aspect that became salient through the work conducted in these classrooms. In this research programme, mental mathematics sessions are designed on a variety of mathematical topics (e.g., algebra, geometry, measurement, statistics, trigonometry, fractions), where students are given short amounts of time to solve tasks given orally and/or on the board, without the use of paper-and-pencil or any material aids. Whereas the central objectives centers on inquiring into the nature of the strategies students engage in to solve the tasks, more seem to be happening in these sessions. In particular, students' solutions and strategies to the task given in the mental mathematics context led to numerous questions, discussions, follow-up explorations, and so forth, by students, which in turn enabled the emergence of significant mathematical issues. This raised interest in investigating these (additional and unplanned) mathematical issues. This represents the core of this lecture, which focuses on the nature of the mathematics (in terms of content and of practices) that frequently unfolds in the mental mathematics sessions conducted. Using an illustrative extract from a mental mathematics session on analytical geometry in a Grade-10 classroom (15-16 years old), the analysis outlines how not only mathematical content is being worked on through these mental mathematics sessions, but also how mathematical practices are being enacted by students. This raises issues about the nature of the environment that these mental mathematics session plunge students into, one that could be tentatively, and boldly, aligned with Papert's concept of mathland.

Keywords: Mental Mathematics, *Didactique des Mathématiques*, Mathematical Practices, Problem Solving, Curriculum Enactment, Geometry and Geometrical and Spatial Thinking

> Being a mathematician is no more definable as knowing a set of mathematical facts than being a poet is definable as knowing a set of linguistic facts. Some modern mathematical education reformers will give this statement a too easy assent with the comment: 'Yes, they must understand, not merely know'. But this misses the capital point that being a mathematician, again like a poet, or a composer, or an engineer, means doing rather than knowing or understanding. (Papert, 1972, p. 249)

Preliminary note: the nature of my research work in *didactique des mathématiques*

My research work is in *didactique des mathématiques*. What does this mean and how does it impact on the nature of the work I conduct? As Douady (1984) expresses, research work in *didactique des mathématiques* investigates the processes and conditions for the production, transformation, communication and acquisition of mathematics, which is not to be reduced to the quest of finding effective teaching methods for mathematical notions. In other words, research work in *didactique des mathématiques* focuses on studying how mathematics happens and advances; which includes its teaching. Brousseau (1991) adds an important aspect to this, mainly that it is a "Science concerned with the production and communication of mathematical knowledge in how these productions and communications are *specific* to mathematics" (*my translation*). What comes out of this is that mathematics and its specificities are central to research work conducted in *didactique des mathématiques*. Hence, questions about mathematics education are addressed *through* mathematics,

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that is, where the *didacticien des mathématiques* is concerned with mathematical experiences and activities in how they are representative, specific and aligned with mathematics themselves. As an example, the interest in problem solving for a *didacticien des mathématiques* is not because doing problem-solving helps learn this or that mathematical concept or because it could contribute to better students' success in mathematics, but mainly because mathematics is defined as a problem-solving endeavor (e.g., Brown & Walter, 2005; Halmos, 1981; Papert, 1972, 1996; Polya, 1957). This is why *didacticiens des mathématiques* undertake studies on problem-solving or argue for its significance: because problem-solving is constitutive of mathematics as a discipline.

Research work in mental mathematics

My research programme is focused on studying mental mathematics in elementary and secondary mathematics classrooms. In this research work, sessions are designed and conducted on a variety of mathematical topics (e.g., algebra, geometry, statistics, measurement, trigonometry, fractions), where classroom students are given short amounts of time to solve tasks given orally and/or on the board, without the use of paper-and-pencil.

Mental mathematics can be defined along the existing research literature, e.g., following Hazekamp's (1986) view, as the solving of mathematical tasks through mental processes without paper-and-pencil or other material aids available. To this one can add that there are frequently time constraints to producing an answer, as well as the fact that questions are often asked orally. The mental mathematics sessions conducted usually follow the same structure, similar to what Douady (1994) suggests by carefully establishing a respectful climate that ensures thay students' share and listen to solutions:

(1) A task is offered orally or on the board;
(2) Students listen and solve the task mentally;
(3) When time is up, students are asked to explain their answer (adequate or not) in detail to the classroom, taken in note on the board (and in some cases students themselves come to the board to explain it);
(4) Other students who solved differently (or thought of solving differently) are invited to offer their answers; once all is said and done, another task is given.

It is often reported that the strategies used to solve mental mathematics tasks differ from those usually referred to in a paper-and-pencil context. Butlen and Pézard (1992), for example, report that the practice of mental mathematics can enable students to develop new and economical ways of solving arithmetic problems that traditional paper-and-pencil contexts rarely afford, because the latter are often focused on techniques that are too time-consuming for a mental mathematics context. These economical ways of solving are said to have the potential to open varied and alternative mathematical routes for handling the concepts under study (e.g., Alain, 1932; Murphy, 2004; Plunkett, 1979; Reys & Nohda, 1994; see also Proulx, 2019). Thus the central objectives of this research on mental mathematics is to inquire into the nature of the mathematical activities (strategies, ways of solving, ideas, reasoning, etc.) that students engage in to solve these tasks.

This said, as these mental mathematics sessions were conducted in classrooms, it became quite apparent that much more than strategies and solutions was happening in these sessions. In effect, the answers given by students and the strategy shared to arrive at them becomes some kind of natural occasion for other students to question or comment them, if they are not convinced or do not understand them. This leads to numerous interactions between students and the Principal Investigator (PI) (and the regular classroom teacher), where students ask important questions about the mathematics at play, which in turn would often lead students to engage in subsequent investigations about these issues (through questions, discussions, follow-up explorations, etc.; see also Cobb et al., 1994, on this). In addition, the sharing of numerous strategies leads invariably to discussions about these strategies, where the various strategies and their answers are compared and discussed by the PI and students concerning their effectiveness, links, (dis-)advantages, possible extensions to other tasks, and so forth. Even if from the outset this was not the scientific objective of the research, this phenomenon became intriguing. And from a *didactique des mathématiques* orientation, some attention was given to how all this was contributing to the advancement of mathematics with students.

Advancement of mathematics: content and practices

The advancement of mathematics can be addressed along two dimensions. The first is relative to the advancement of *mathematical content*. Mathematics is filed with content, from number to geometry and algebra, to name a few, through various algorithms, formulas, procedures, methods, definitions, theories and theorems about them. Analyses of the advancement of content in classrooms focuses on the development of this content with students, that is, on their understandings and reasoning relative to this mathematical content. Having said this, as Papert's above quote insists, mathematics is not only about its content; it is an activity that is done and takes shape in action (see also Brown & Walter, 2005; Hersh, 2014; Lockhart, 2009; Schoenfeld, 2020). Mathematics is about doing mathematics; mathematics is a practice. Another dimension thus concerns *mathematical practices*. This second dimension of the advancement of mathematics in classrooms is about the development of mathematical practices in students, that is, how these emerge, unfold, progress, and so forth, as mathematics is being explored and produced.

In other words, mathematics is composed of content and practices, where this mathematical content is explored and engaged with. Intertwined with the advancement of content, the emergence and development of mathematical practices thus acts as a fundamental dimension to consider in relation to mathematics. It is also along these lines that Lampert (1990a) raises the relevance of working on a double agenda, that is, simultaneously on *of* and *about* mathematics:

> This meant that I needed to work on two teaching agendas simultaneously. One agenda was related to the goal of students' acquiring technical skills and knowledge in the discipline, which could be called knowledge of mathematics, or mathematical content. The other agenda, of course, was working toward the goal of students' acquiring the skills and disposition necessary to participate in disciplinary discourses, which could be called knowledge about mathematics, or mathematical practice. (p. 44)

Both these dimensions of content and practices have been salient in the mental mathematics sessions conducted. This research, strongly grounded in Papert's work (e.g. 1972, 1980, 1993, 1996; see also Barabé & Proulx, 2017), compelled investigations of mathematical practices. Papert is indeed quite adamant on the importance of the development of mathematical practices, where mathematics is not something given and fixed, but is alive and a source of ongoing investigations in order to enrich students' experiences and culture in mathematics (see, e.g., 1993, 1996). This idea also relates to Bauersfeld's (1995, 1998) notion of plunging students into a "culture of mathematizing", where mathematical practices unfold and take shape through interactions and investigations.

Participants in a culture of mathematizing are seen as authors and producers of mathematical knowledge, understandings and meanings. In the establishment and development of such a culture, where mathematical practices unfold and concepts and methods are explored and worked on, students are encouraged to generate ideas, questions and problems, to make explicit and share understandings and solutions, to develop explanations and argumentations to support the solutions and strategies put forth, to negotiate proposed meanings, to share and explore various ways of understanding problems, concepts, symbolism, and representations, and to assess and validate other's understandings and ways of doing (see e.g. Bartolini Bussi, 1998; Bednarz, 1998; Borasi, 1992, 1996; Brown & Walter, 2005; Cobb & Yackel, 1998; Lampert, 1990a; Schoenfeld, 2020; Voigt, 1985, 1994). From these practices, a number of elements can be outlined to characterize and analyze the advancement of mathematics.

- *The emergence of a community of validation*. Central to a mathematics-producing practice are participants who are engaged in explaining, discussing, arguing, and validating mathematical understandings and meanings (Boaler, 1999; Borasi, 1992; Hersh, 1997; Krummheuer, 1995; Lakatos, 1976; Lampert, 1990a).
- *The role, relevance and development of mathematical languages, symbolisms and conventions*. Mathematical symbolism, languages and conventions, and their development, are used to express mathematical understandings, explanations, arguments, etc., and play a major role in the emergence of mathematics and mathematical thinking (Bednarz et al., 1993; Byers & Erlwanger, 1984; Byers & Herscovics, 1977; Lampert, 1990b; Lockhart, 2017).
- *The role given to errors and how they are handled*. Errors play and have played a fundamental role in the emergence of mathematical thinking and understanding. The way they have been handled has enable new ways of seeing and understanding mathematics, leading to unpredicted or as yet not thought of avenues (Borasi, 1996; Hadamard, 1945).
- *The solving and posing of problems*. Doing mathematics is an activity of posing and solving problems of many kinds (Bkouche, Charlot & Rouche, 1992; Brown & Walter, 2005; Hersh, 1997; Lang, 1985; Polya, 1945), where explorations of mathematical content have contributed to the development of additional mathematical content.
- *The authorship, ownership and responsibility in mathematics*. Doing mathematics imposes an active engagement. People doing mathematics do not conceive of themselves as mere consumers or receivers of mathematics, but as producers and even authors of mathematics (Papert, 1996; Povey & Burton, 1999; Schoenfeld, 1994). Mathematics confers a double sense of responsibilities (Borasi, 1992, 1996), where people doing mathematics are responsible for the mathematics they produce and also responsible for producing mathematics.

As scientific interest arose about these dimensions relative to the advancement of mathematics in the mental mathematics sessions conducted, the following question oriented the inquiry: *In what ways is mathematics advancing in the mental mathematics sessions, under both its mathematical content and practices dimensions?* As a way of showing how the advancement of mathematics happened in the sessions, an extract taken from one session is presented. This extract is then looked into in relation to how mathematics content and practices advance, as a way of offering an initial illustration of what it can mean to analyze the advancement of mathematics in these mental mathematics sessions.

Extract from a mental mathematics session

The extract is taken from a session led by the PI in a Grade-10 classroom of about 30 students, who were working on analytical geometry in relation to distances (points, midpoints, lines, etc.) and had been initiated to usual algebraic formulas. One of the tasks given to students was "Find the distance between (0,0) and (4,3) in the plane" (given orally, with points drawn on a Cartesian plane on the front board); they had 15 seconds to answer without recourse to paper and pencil or any other material. When time was up, students were invited to share and justify their solutions to the group. The following is a synthesis of the strategies engaged in and the discussions, questions, and explorations that ensued.

The first strategy referred to applying the usual distance formula (D=$\sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}$), leading to 5 as a distance. A second strategy suggested drawing a triangle in the plane, with sides 3 and 4, for then finding the hypotenuse by using Pythagoras (Figure 1a).

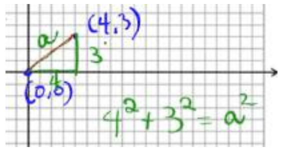


Figure 1a – Drawing the right triangle

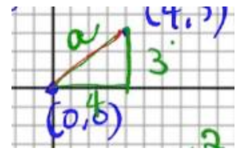


Figure 1b – Close-up on the triangle

Another student then suggested a third strategy, coming to the board to trace a red segment to count directly on it from (0,0) to (4,3) as in Figure 1b. Starting from (0,0), she counted "the number of points" to arrive at (4,3), counting the number of whole-number coordinate points from (0,0) to (4,3). While doing this at the board, she suddenly stopped and mentioned that her red segment did not go through the points she envisaged, which made the counting difficult. The PI then traced another segment going through square diagonals linking two separate points, which could enable counting the number of (whole-number) coordinate points from one point to the next (giving 4 as a distance, Figure 2). The student agreed that for this case, it would work.

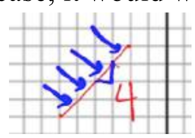
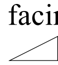


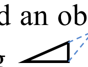
Figure 2 – Line drawn through square diagonals

The PI then asked if the measure obtained with square diagonal lengths was identical to that obtained with the side of the square (drawing  on the board).

> One student asserted that both lengths were not identical, because the diagonal of the square was not of the same length as the square's side. Another explained that both lengths were different, because the hypotenuse is always the longer side in a triangle. Finally, a student claimed that the diagonal was longer because it faces the wider angle.

The PI then asked if that last assertion about facing the wider angle was always true, and if so why (drawing on the board a random right triangle ).

> One student, pointing at the triangle, stated that it was indeed the case in this drawn triangle. Another student explained that in a triangle the bigger the angle the longer the opposite side, mentioning that if the side-hypotenuse had been longer, the opposite angle would have been wider. And, because the sum of the (measures of the) angles in a triangle is 180^o, then the 90^o angle is always the wider one, the other 90^o being shared between the remaining two angles.

Using the drawing of the triangle, the PI simulated the variation of the right angle toward an obtuse one and traced the resulting side obtained, showing how it would become longer (drawing  on the board). He then moved it toward producing an acute angle, asking students if their "theory" about opposite side of the angle worked for any angle, like acute ones.

> One student asserted that it works for isosceles triangles, with equal sides facing equal angles, and another mentioned that it is the same for the equilateral triangle, because it is "everywhere the same" with same angles and same side lengths.

The PI explained that these ideas about the diagonals being longer than the side underlined the fact that this initial strategy amounted to counting diagonals, that is, the number of diagonals of a unit square. And, that this offered a different sort of measure for the (same) distance between the two points: one in terms of units and one in terms of diagonals. A student added that if one knows the value of the diagonal (e.g. 1.2 or else), then one could find the number of unit squares for the diagonal-segment by multiplying by that factor.

One student offered a fourth strategy to find the distance, suggesting using the sine law with angles of 45^o. The PI asked the student how he knew that both angles were 45^o in the triangle. As skepticism grew in the classroom, the PI suggested that students inquire, in small groups or individually, if the triangle's angle were 45^o or not, and to be able to convince others. After 5-6 minutes of exploration, students were invited to share their findings.

> One student explained that on her exam checklist there is an isosceles right triangle with 45^o angles. Thus with this triangle of side length of 4 and 3, one cannot directly assert that it is 45^o because it is not an isosceles triangle as its sides are not equal. Another student illustrated on the board that if one "completes" the initial triangle into a rectangle ( , see Figure 3a), then the hypotenuses of both triangles are the rectangle's diagonal which cuts it in two equal parts and thus cuts its angle in two equal 45^o parts.

As the PI highlighted that the two arguments were opposed, one student replied not agreeing with the last argument, drawing on the board a random rectangle with its diagonal (Figure 3b), and asserting that in this rectangle *it was not certain* that the angle was divided into two equal parts. Another student added that because the sides of the triangle were not identical (of 3 and 4), then the diagonal would not necessarily cut the 90^o angle in two equal parts of 45^o.

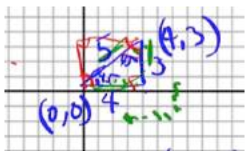


Figure 3a – The "completed" rectangle

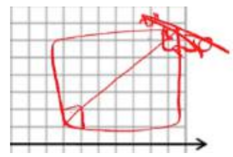


Figure 3b – The "counter" rectangle

The PI highlighted that this last argument reused aspects of the precedent "theory" that the longer side faces the wider angle in the triangle. Hence, following this, a longer side needed to face a wider angle. Then a counter-example was offered to the group.

> The student who made reference to the checklist asserted that sometimes in their exams right triangles did not have 45^o angles, for example, one with 32^o and 58^o; coming to the board to draw it (Figure 4). She completed her drawing to create a rectangle, explaining that the diagonal cuts as well this rectangle in two parts, but that the angles obtained are not of 45^o.

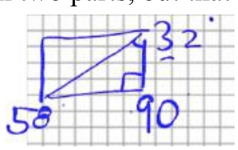


Figure 4 – The triangle counter-example with angles of 32^o and 58^o, and the rectangle

The PI asserted that this offered a counter-example, with a type of right triangle frequently met that did not have angles of 45^o.

> One student added that because all sides were different, then their associated angles would be different, the longer side needed to face a wider angle, which would lead to different angles.

The PI then highlighted the work of one student who drew a square in his notebook to assess the 45^o situation. Drawing a triangle of sides 3-4-5, he extended the cathetus of 3 toward one of 4 to create a 4x4 square. Then, because in the previous unit-square the angles were of 45^o, in this 4x4 they were 45^o as well (Figure 5). Comparing hypotenuses of both triangles, it illustrated that in the initial 3-4-5 right triangle, the angle is smaller than the right triangle of side 4 and 4. All this led students to appear to agree that the angle was not 45^o, ending the explorations (and leading the PI to offer another task for the students to solve).

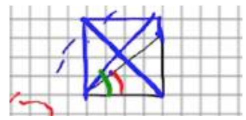
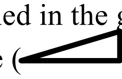


Figure 5 – Comparing triangles within a square

Analysis of the advancement of mathematics

A *didactique des mathématiques* analysis of this extract in relation to the advancement of mathematics underlines issues of mathematical content and of mathematical practices. First, mathematical content is significantly present in this extract through the explorations undertaken. Some mathematical content is engaged with more superficially or in an isolated way, without requiring subsequent exploring and mostly being referred to: Pythagoras' relation, distance formula, hypothenuse, angles (acute, obtuse, right), triangles (various types, and isosceles and equilateral). These are not explored in depth, but are mobilized during the session and play an important part in it. Other mathematics content takes a more important place, enabling or representing some mathematical advances in the session through deeper explorations than the former: the sum of the measures of the angles of a triangle is 180^{o}, the possibility of having two different measures for the same distance, the relationship between the rectangle's diagonal, and the bisector of its angles. Finally, some content appears at the heart of the explorations in the session, thought of and recurrently being engaged with by students: the difference between the (measure of the) square diagonal and (the measure of) its side, and the relation between and variation of one side of the triangle and its opposite angle. There would obviously be more to outline, and along much subtler lines, but what is significant is the magnitude of the mathematical content worked on, mobilized, and continually explored with the students.

Second, students are enacting a variety mathematical practices, which participate in the environment where the mathematical content is taking shape. In sum, the mathematical contents engaged with in the session are grounded in these mathematical practices:

- *The emergence of a community of validation*. The investigation of the 45^{o} angle is an example of how a community of validation was established, in which students offered conjectures, argued and counter-argued on the ideas suggested, justified their claim, developed elements to prove it, engaged in reflections to establish what works and does not, and why, etc. The mathematical "truths" were not passively received from outside, from an external authority, but were debated and worked on to develop consensus.
- *The role, relevance and development of mathematical languages, symbolisms and conventions*. Although complex to analyse from a short extract, it is possible to seize some of the symbolisms and representations that took shape in it. For example, the manner of drawing rectangles and triangles with a "cut" to argue about the value of their angles is representative of a strong symbolization that became established in the group, that evolved, and that was used throughout the session. Thus, from a triangle (), students were led to "complete" it to make a rectangle (), enabling them to discuss and explore what happens with the rectangle's and triangle's angles. It is this specific symbolic representation that is used in Figures 3b and 4 to argue and counter-argue about the rectangle's diagonal and the division in half of the 90^{o} angle. This "invented" representation to symbolize the relationship between rectangles and triangles regarding their angles contributed to the mathematical understandings, and was often reused by students in the session.
- *The role given to errors and how they are handled*. Errors have played a productive role in the session, provoking additional questions and explorations. For example, the third strategy about measuring the distance between the points through the diagonal of the unit-square has

unleashed important questioning on the difference between the diagonal and the side of the square, and has led to the idea that it is possible to have different measures for the same distance. The suggestion that the triangle had a 45^o angle also provoked the investigation about triangles' angles and sides, as well as rectangles' sides and diagonals. None of these assertions, even when erroneous, were criticized and all were taken seriously: they were respected as authentic mathematical productions and enabled deeper understandings of the mathematics at play.

- *The solving and posing of problems*. Throughout the session, questions were asked and sub-problems emerged, unpredicted and contingent on the ongoing explorations undertaken (e.g. diagonal of the unit-square; the 45^o angle; the diagonal splitting the 90^o angle into two equal parts). Students raised and engaged intensely in these questions and sub-problems. It is through these questions and problems that the main part, if not the entirety, of the mathematical content was explored and deepened.
- *The authorship, ownership and responsibility in mathematics*. Students took an active part in the investigation through a number of mathematical assertions and proposals (through strategies, answers, questions, disagreement, explanations, etc.). In this sense, they took ownership of the ideas produced and were engaged in producing them. This double-responsibility took place as students were not passive in the session, but contributed to it with their own ideas. As an example, students' spontaneous use of the front board shows how they felt compelled to share their ideas and participate in the explorations to reach a consensus: they show ownership over this consensus and do not appear to wait for someone else to reach it for them, interacting with others and the PI, raising issues, arguing, questioning, responding, etc.

Another mathematical practice also comes out of this extract, and one considered of significance in mathematics. It is related to what Papert (1980, 1993) calls *theorizing*. In the discussions about the difference between the measure of the diagonal and the side of the square, an important theory was suggested by students: the bigger side of the triangle faces its bigger angle. First, this theory was mainly an assertion, some sort of conjecture. But, after some questions raised by the PI (*Does it work all the time? / What happens if the angle changes? / etc.*), it was increasingly confirmed by and through students' justifications. This theory was then used by others, and as much by the students than by the PI, to address the issues about the 45^o angle: if the measure of one side of the triangle is not the same as another, then neither can be the opposite! Throughout the session, this theory took shape and strengthened, giving rise to a number of side assertions, in the form of corollaries, like the following:

Corollary 1: In a triangle, the smallest angle is always opposed to the smallest side.
Corollary 2: In a triangle, the smaller an angle is, the smaller its opposite side is.
Corollary 3: In an isosceles triangle, both equal angles are opposed to both equal sides.
Corollary 4: In an equilateral triangle, angles are the same, linked to sides of same length.
Corollary 5: Since the sides are not equal, its angles are not equal either.
Corollary 6: Since the sides are of different length, they opposed angles of different size.

And the list could go on. Without always being stated explicitly, the arguments and explanations related to the initial theory, that justified it, underlined these ideas and strengthened them. This made the theory increasingly accepted by students and the PI, to the point of being used itself as an argument. It is in this sense that this theory, and its corollaries, became established during the session, and became "proven". It can be seen as some kind of proof by use, which is shown to be truthful through its efficient functionality and recurrence (Hersh, 2014). *The proof of the pudding is in the eating!* The establishment of theories thus acts here as an additional mathematical practice being put forth in the session.

Concluding remarks

The above analysis could be deepened and refined. However this sketch, albeit rapid, is significant: it illustrates how mathematics not only advanced in relation to its content, but also relative to its practices, and how both content and practices are intermingled in this advancement, going hand in hand, participating in the unfolding of the other. Content arises through mathematical practices, which in turn are geared toward specific contents. The need to talk about triangles and their angles as content gave rise to a specific *symbolisation* to represent it, which in turn helped to make sense of triangles, rectangle and their angles. The need to understand the 45^{o} angle as content, and the skepticism that it caused, led the *community of validation* to take shape, helping in return to give stronger meaning to the 45^{o} angle. The notion of the measure of the side of a square and its diagonal made emerge a question about their difference, becoming a *sub-problem* to inquire into, which led not only to understandings about their difference in the square, but also gave rise to the *theory* of the triangle's angle and its relation to its opposite side. And the list could go on, for each dimension of mathematical practices outlined, each linked to aspects of mathematical content covered in the session.

This extract is only a short glimpse into the nature of the work conducted regularly with groups like these in mental mathematics settings. As these mathematical practices continually unfolded, sessions after sessions and with different group of students, one cannot but be seized by how students plunged deeply into aspects at the heart of Bauersfeld's (1995, 1998) culture of mathematizing. The mathematical ideas emerge, are alive and flow dynamically. The students are strongly engaged, compelled to contribute, enthusiastic in responding to one another and to the ideas shared, and so forth.

However, above all, this was not staged nor planned. Mental mathematics sessions are usually designed to gather and then analyse students' strategies about various mental mathematics tasks. But classrooms are what they are, and students are who they are: asking them to solve mental mathematics tasks made emerge lots of questioning from them, and between them, about the mathematics. The tasks then became springboards for inquiry or "seeds" for explorations (Borasi, 1992, 1996; Schoenfeld, 2020), as opportunities for developing not only mathematical content but also mathematical practices. This is why the mental mathematics environment that students seem to be plunged into appeared to be worth reflecting on.

Although Papert never profoundly developed this concept, one is compelled to wonder if this environment of exploration happening in the mental mathematics sessions could represent, at least a little, what he had in mind with his *mathland*. Here, for example, is one quote taken from *The Children's machine*:

> It is thoroughly embedded in our culture that some of us have a head for figures while most don't, and accordingly, most people think of themselves as not mathematically minded. But what do we say about children who have trouble learning French in American schools? Whatever the explanation of their difficult, one certainly cannot ascribe it to a lack of aptitude for French – we can be sure that most of these children would have learned French perfectly well had they been born and raised in France. […] In the same way, we have no better reason to suppose that these children who have trouble with math lack mathematical intelligence than to suppose that the others lack "French intelligence". We are left with the question: What would happen if children who can't do math grew up in a Mathland, a place that is to math what France is to French? […] while what happened in the regular math class was more like the learning math as a foreign language. […] In the math class, where knowledge is not used but simply piled up like the bricks forming a dead building, there is no room for significant experimenting. (Papert, 1993, p. 64)

However bold, asserting that the mental mathematics environment in which students are plunged, for which the Grade-10 extract is an illustration of, could be aligned with a form of *mathland* has a nice ring to it. And, this ring leads one to become attentive to the strength of the engagement and the richness of the explorations undertaken. It seems to orient the focus, as Papert insisted, on doing mathematics more than on knowing mathematics. In this sense, doing mental mathematics becomes more about inquiring than about knowing facts (see PME-NA research report in Proulx, 2014, 2015a; or others e.g. in Proulx 2013, 2015b, 2019).

Although at first a curiosity, inquiring into the environment of the mental mathematics sessions seemed to help draw out both these content and practices dimensions, and their intertwinement in the advancement of mathematics in the sessions. And it might be where Papert's *mathland* fits in well, that is, in an environment where mathematics grows as much in terms of content as in terms of practices.

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